1 Transverse Cooling

1.1 Recap Beam Definitions

1.1.1 Emittance

$$\text{normalized emittance} \;=\; \frac{\text{Phase Space Area}}{\text{m c}}$$

If x and p_x both Gaussian and uncorrelated, then area is that of an upright ellipse

$$\epsilon_\perp = \frac{\sigma_{p_\perp}\sigma_x}{mc} = \sigma_\theta \sigma_x \quad (\gamma\beta_v) \qquad\qquad (\pi\ m\ rad)$$

$$\epsilon_\parallel = \frac{\sigma_{p_\parallel}\sigma_z}{mc} = \frac{\sigma_p}{p}\ \sigma_z \quad (\gamma\beta_v) \qquad\qquad (\pi\ m\ rad)$$

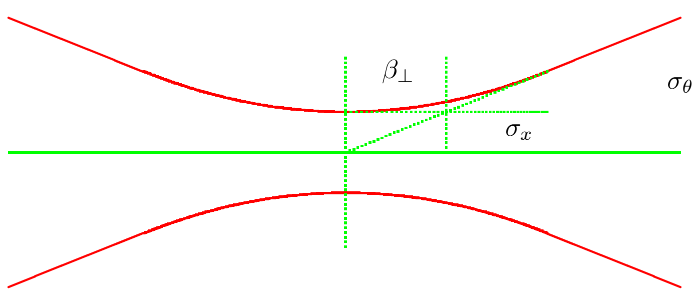
$$\epsilon_6 = \epsilon_\perp^2\ \ \epsilon_\parallel \qquad\qquad (\pi\ m)^3$$

Note that, by convention, the π is not included in the calculated values, but added to the dimension

1.1.2 Beta$_{Courant-Schneider}$

Again upright ellipse, e.g. at Focus:

$$\beta_\perp \;=\; \frac{\sigma_x}{\sigma_\theta}$$



Then, using emittance definition:

$$\sigma_x \;=\; \sqrt{\epsilon_\perp\ \beta_\perp\ \frac{1}{\beta_v\gamma}}$$

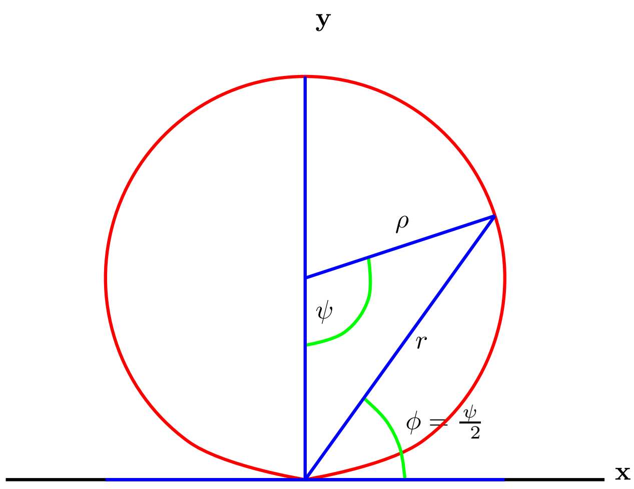
$$\sigma_\theta \ = \sqrt{\frac{\epsilon_\perp}{\beta_\perp} \ \frac{1}{\beta_v \gamma}}$$

$\beta_\perp$ is defined by the beam, but a lattice can have a β_o that "matches" a beam with that $\beta_\perp$

1.2 Introduction to Solenoid Focussing

Motion in Long Solenoid

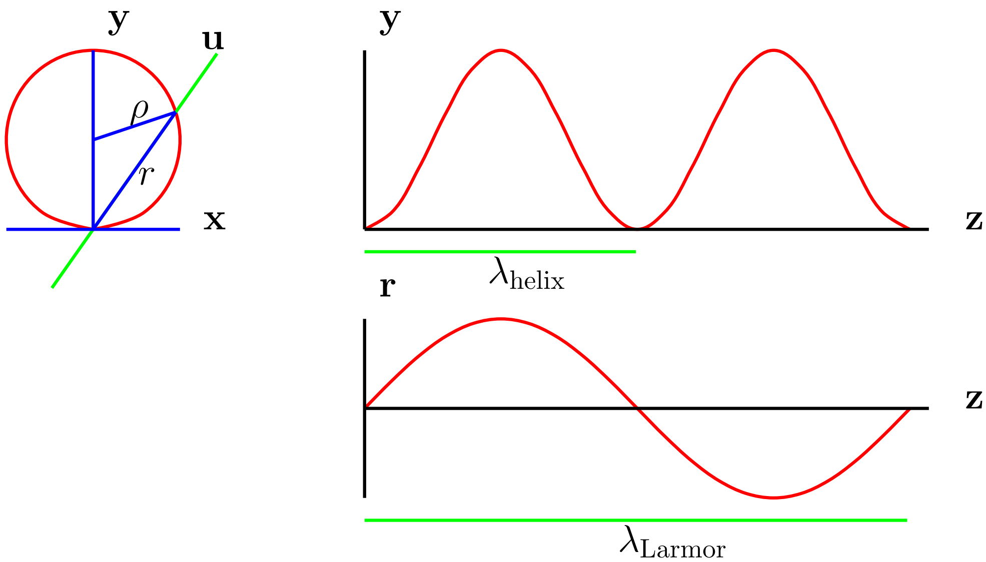
$$\rho \ = \ \frac{p_\perp}{c \ B_z}$$



$$x \ = \ \rho \sin(\psi)$$

$$y \ = \ \rho \ (1 - \cos(\psi))$$

Larmor Plane



$$
\begin{aligned}
r &= |2\rho\ (\sin(\phi))| \\
u &= 2\rho\ (\sin(\phi)) \\
\lambda_{\text{Helix}} &= 2\pi\ \frac{p_z}{c\ B_z} \\
\lambda_{\text{Larmor}} &= 2\pi\ \frac{p_z}{2\ c\ B_z}
\end{aligned}
$$

Ted told us that there is a

$$\beta = \frac{d\phi}{dx}$$

which gives

$$\beta_{\text{Ted}} = \frac{\lambda_{\text{Larmor}}}{2\pi}$$

Phase Ellipse

$$
\begin{aligned}
\beta_{\perp} &= \frac{\text{maximum of } u}{\text{maximum of } du/dz} \\
u &= 2\rho\ \sin(\phi) \\
\frac{du}{dz} &= \frac{2\rho\ cos(\phi)}{\beta_{\text{Ted}}}
\end{aligned}
$$

so

$$\beta_{\perp} = \beta_{\text{Ted}}$$

Focussing "Force"

$$\frac{d^2u}{dz^2} \;=\; -k\;u$$

$$k \;=\; \left(\frac{c\;B_z}{2\;p_z}\right)^2$$

TRUE FOR VARYING FIELDS TOO

Note: the focusing "Force" $\propto\; B_z^2$ and $\propto\; 1/p_z^2$

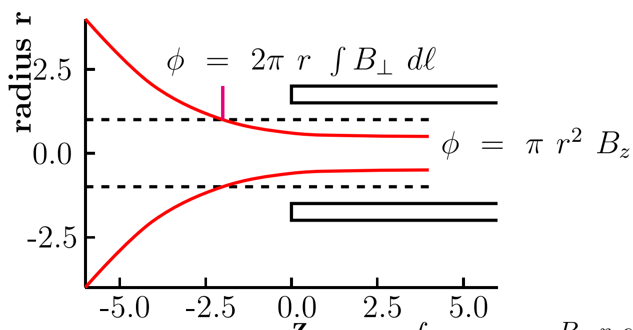
Is not good for high p, but beats quads at low p, and focuses in both directions simultaneously.

Angular Momentum

$$M \;=\; r\;p_\perp\;\sin(\phi)$$

$$M \;=\; \frac{r^2\;B_z\;c}{2}$$

Entering a solenoid



$$\Delta p_\perp \;=\; \int B_r\;dz \;=\; \frac{B_z\;r\;c}{2}$$

$$M \;=\; p_\perp\;r \;=\; \frac{r^2\;B_z\;c}{2}$$

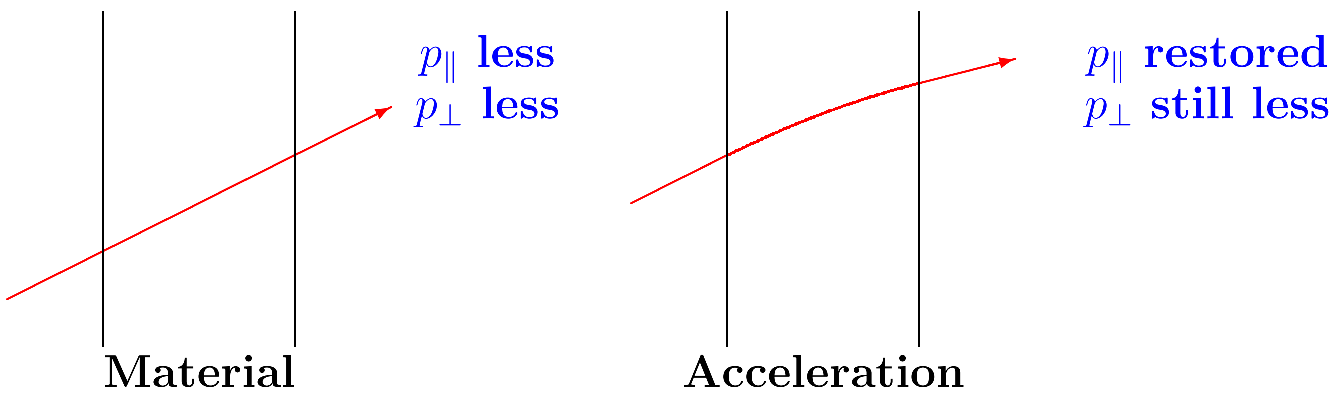
SAME AS ABOVE

In fact, if outside B, $M = 0$, then eqn. is true INDEPENDENT OF EARLIER B's SHAPE

If there is M_o outside B (known as Canonical Angular Momentum), then in B_z:

$$M \;=\; M_o \;+\; p_\perp\; r \;=\; \frac{r^2\;B_z\;c}{2}$$

1.3 Transverse Cooling



1.3.1 Cooling rate vs. Energy

$$\epsilon_{x,y} \;=\; \gamma\beta_v\;\sigma_\theta\;\sigma_{x,y} \tag{1}$$

If there is no Coulomb scattering, or other sources of emittance heating, then σ_θ and $\sigma_{x,y}$ are unchanged by energy loss, but p and thus $\beta\gamma$ is reduced. So the fractional cooling $d\epsilon\;/\epsilon$ is:

$$\frac{d\epsilon}{\epsilon} \;=\; \frac{dp}{p} \;=\; \frac{dE}{E}\;\frac{1}{\beta_v^2} \tag{2}$$

which, for a given energy change, strongly favors cooling at low energy.

But if total acceleration were not important, e.g. if the cooling is done in a ring, then there is another criterion: The cooling per fractional loss of particles by decay:

$$Q \;=\; \frac{d\epsilon/\epsilon}{dn/n} \;=\; \frac{dp/p}{d\ell/c\beta_v\gamma\tau}$$

$$=\; \frac{dE/E\;\;1/\beta_v^2}{d\ell/(c\gamma\beta_v\tau)}$$

$$= \ (c\tau/m_\mu) \ \ \frac{dE}{d\ell} \ \ \frac{1}{\beta_v}$$

Which only mildly favours low energy

1.3.2 Heating Terms

$$\epsilon_{x,y} \ = \ \gamma\beta_v \ \sigma_\theta \ \sigma_{x,y} \tag{3}$$

Between scatters the drift conserves emittance (Liouiville).

When there is scattering, $\sigma_{x,y}$ is conserved, but σ_θ is increased.

$$\begin{aligned}\Delta(\epsilon_{x,y})^2 \ &= \ \gamma^2\beta_v^2 \ \sigma_{x,y}^2\Delta(\sigma_\theta^2) \\ 2\epsilon \ \Delta\epsilon \ &= \ \gamma^2\beta_v^2 \left(\frac{\epsilon\beta_\perp}{\gamma\beta_v}\right) \ \Delta(\sigma_\theta^2) \\ \Delta\epsilon \ &= \ \frac{\beta_\perp\gamma\beta_v}{2} \ \Delta(\sigma_\theta^2)\end{aligned}$$

e.g. from Particle data booklet

$$\Delta(\sigma_\theta^2) \approx \left(\frac{14.1 \ 10^6}{(p)\beta_v}\right)^2 \ \frac{\Delta s}{L_R}$$

$$\Delta\epsilon \ = \ \frac{\beta_\perp}{\gamma\beta_v^3} \ dE \ \ \left(\left(\frac{14.1 \ 10^6}{2(m_\mu)}\right)^2 \ \frac{1}{L_R\Delta E/ds}\right)$$

Defining

$$C(mat, E) \ = \ \frac{1}{2} \ \left(\frac{14.1 \ 10^6}{(m_\mu)}\right)^2 \ \frac{1}{L_R \ d\gamma/ds} \tag{4}$$

then

$$\frac{\Delta\epsilon}{\epsilon} \ = \ dE \ \ \frac{\beta_\perp}{\epsilon\gamma\beta_v^3} \ \ C(mat, E) \tag{5}$$

Equating this with equation 2

$$dE \ \frac{1}{\beta_v^2 \ E} \ = \ dE \ \ \frac{\beta_\perp}{\epsilon\gamma\beta_v^3} \ \ C(mat, E)$$

gives the equilibrium emittance ϵ_o:

$$\epsilon_{x,y}(min) \ \ = \ \frac{\beta_\perp}{\beta_v} \ \ C(mat, E) \tag{6}$$

Rate of Cooling

$$\frac{d\epsilon}{\epsilon} \ = \ \left(1 \ - \ \frac{\epsilon_{\min}}{\epsilon}\right) \ \frac{dp}{p} \tag{7}$$

At energies such as to give minimum ionization loss, the constant C_o for various materials are approximately:

material	T ^{o}K	density kg/m^3	dE/dx MeV/m	L_R m	C_o 10^{-4}
Liquid H_2	20	71	28.7	8.65	38
Liquid He	4	125	24.2	7.55	51
LiH	300	820	159	0.971	61
Li	300	530	87.5	1.55	69
Be	300	1850	295	0.353	89
Al	300	2700	436	0.089	248

Clearly Liquid Hydrogen is far the best material, but has cryogenic and safety complications, and requires windows made of Aluminum or other material which will significantly degrade the performance.

1.3.3 Beam Divergence Angles

$$\sigma_\theta = \sqrt{\frac{\epsilon_\perp}{\beta_\perp \ \beta_v \gamma}}$$

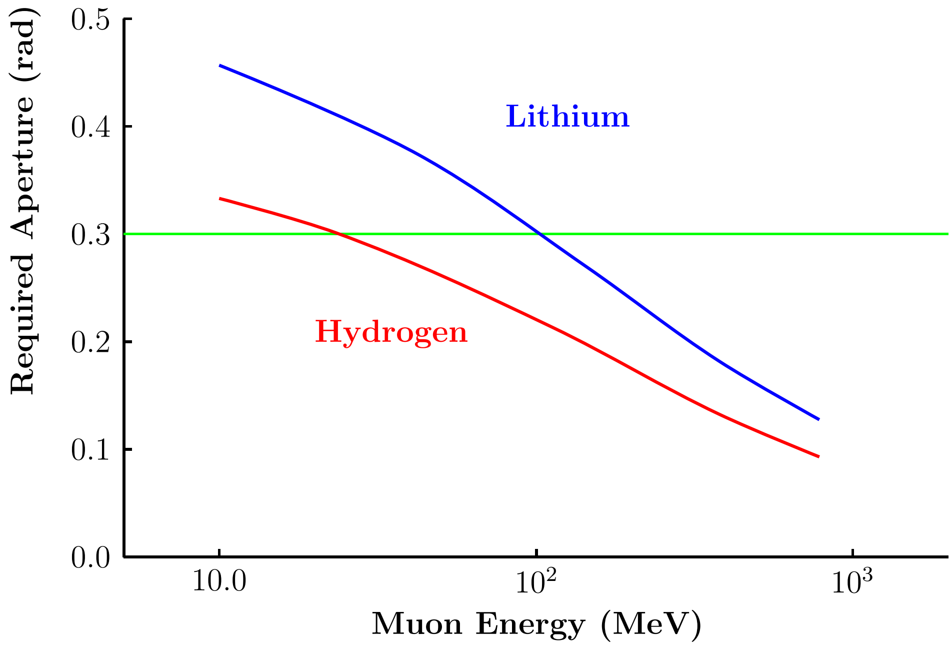
so, from equation 6, for a beam in equilibrium

$$\sigma_\theta = \sqrt{\frac{C(mat,E)}{\beta_v^2 \gamma}}$$

and for 50 % of maximum cooling and an aperture at 3 σ, the aperture $\mathcal{A}$ of the system must be

$$\mathcal{A} = 3\sqrt{2} \sqrt{\frac{C(mat,E)}{\beta_v^2 \gamma}} \tag{8}$$

Apertures for hydrogen and lithium are plotted vs. energy below. These are very large angles, and if we limit apertures to less than 0.3, then this requirement sets lower energy limits of about 100 MeV ($\approx$ 170 MeV/c) for Lithium, and about 25 MeV ($\approx$ 75 MeV/c) for hydrogen.



1.4 Focusing Systems

1.4.1 Solenoid

***

In a solenoid with axial field B_{sol}

$$\beta_\perp \;=\; \frac{2\;(p)}{c\;B_{sol}}$$

so

$$\epsilon_{x,y}(min) \;=\; C(mat,E)\;\frac{2\;\gamma\;(m_\mu)}{B_{sol}\;c} \tag{9}$$

For $E = 100\ MeV$ $(p \approx 170\ MeV/c)$, $B = 20\ T$, then $\beta \approx 5.7\ cm.$ and

$\epsilon_{x,y} \approx 266(\pi mm\ mrad)$.

1.4.2 Current Carrying Rod

In a rod carrying a uniform axial current, the azimuthal magnetic field B varies linearly with the radius r.

A muon traveling down it:

$$\frac{d^2r}{dr^2} = \frac{B\ c}{(p)} = \frac{r\ c}{(p)}\ \frac{dB}{dr}$$

so orbits oscillate with

$$\beta_\perp^2 = \frac{\gamma\beta_v}{dB/dr}\ \frac{(m_\mu)}{c} \tag{10}$$

If we set the rod radius a to be f_{ap} times the rms beam size $\sigma_{x,y}$,

$$\sigma_{x,y} = \sqrt{\frac{\epsilon_{x,y}\ \beta_\perp}{\beta_v\gamma}}$$

and if the field at the surface is B_{max}, then

$$\beta_\perp^2 = \frac{\gamma\beta_v(m_\mu)\ f_{ap}}{B_{max}\ c}\ \sqrt{\frac{\epsilon_{x,y}\ \beta}{\gamma\ \beta_v}}$$

from which we get:

$$\beta_\perp = \left(\frac{f_{ap}\ (m_\mu)}{B_{max}\ c}\right)^{2/3}\ (\gamma\beta_v\ \epsilon_{x,y})^{1/3}$$

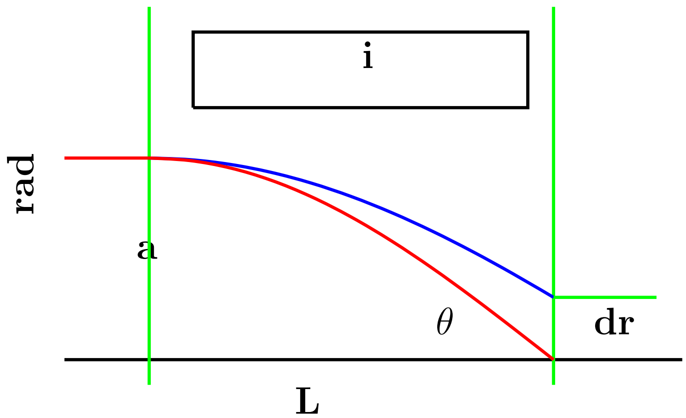
puting this in equation 6

$$\epsilon_{x,y}(min) = (C(mat,E))^{1.5}\left(\frac{f_{ap}\ (m_\mu)}{B_{max}\ c\ \beta_v}\right)\ \sqrt{\gamma} \tag{11}$$

e.g. B_{max}=10 T, f_{ap}=3, E=100 MeV, then $\beta_\perp = 1.23$ cm, and

$\epsilon(min)$=100 (π mm mrad)

1.4.3 At a Focus



The minimum beta obtainable at a focus is set by chromatic abberations, i.e. momentum dependent effects. Assuming no external correction:

$$\beta(min) \;=\; \frac{dr}{\theta} \;=\; \frac{a}{\theta}\; dp/p \;=\; L\; dp/p$$

For a solenoid with axial filed B, and momentum p

$$L \;=\; \frac{\pi}{2}\;\beta_o \;=\; \frac{\pi\;(p)}{c\;B}$$

so

$$\beta(min) \;=\; \left(\frac{\pi\;(p)}{c\;B}\right)\;\frac{dp}{p}$$

$$\epsilon(min) \;=\; C_{H_2}\;\left(\frac{\pi(E)}{c\;B}\right)\;\frac{dp}{p}$$

e.g. p=.17 MeV, B=5 T, dp/p=5%, $\beta(min)$ = 1.8 cm, and

$\epsilon(min)$ = 82 π mm mrad

But as p falls, the possible coil thickness also falls. Below some mom we may have to fix the current density i:

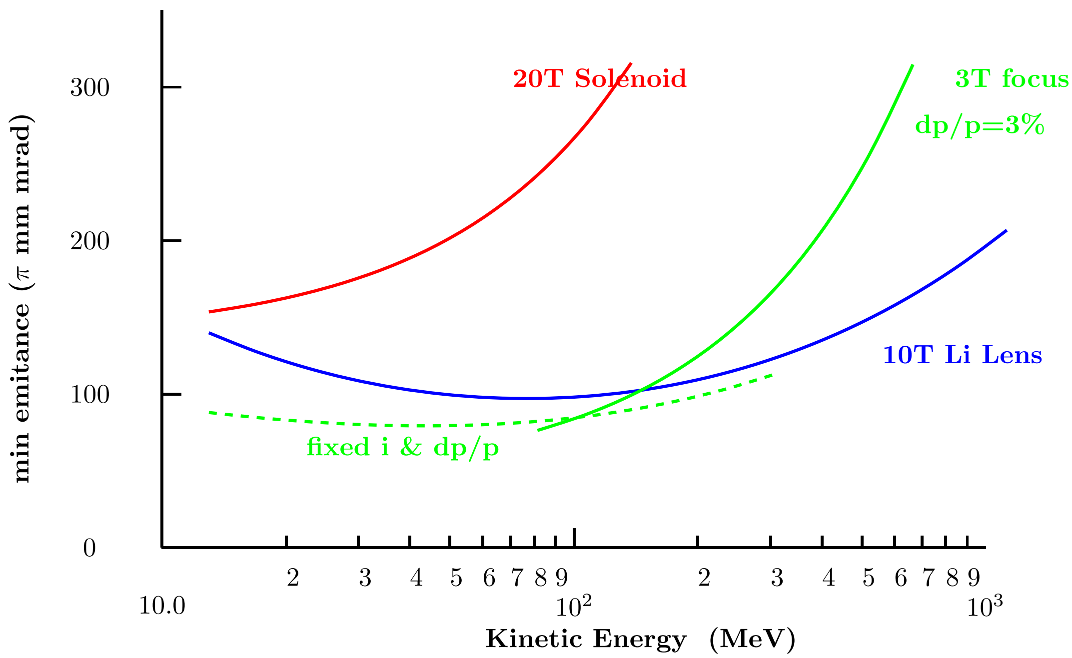
$$B \;\propto\; \frac{p}{B}$$

and so

$$\beta(min) \propto \sqrt{\frac{\gamma}{\beta}}$$

1.4.4 Compare Focusing

Assuming that the current limits the focus beta below 100 MeV, then we can compare the methods as a function of the beam kinetic energy.



We see that, for the parameters selected, no method allows transverse cooling below about 80 (π mm mrad)

1.5 Simulation

- Calculations assume Gaussian scatter and straggling, and small angles, and thus approximate.
- Accurate results require simulation
- Several "local" codes
  Two Documented codes:

GEANT & ICOOL
Both have:

- Global fields
  unlike MAD, TRANSORT etc.
- Choices of scattering and straggling formulations
- Standing Wave RF fields
- allow use of both
  1. Maxwellian, or
  2. "hard edged" magnetic fields
- flexible Geometries
- Good tracking

The differences in handling bends discussed in section **??**

1.6 Angular Momentum Problem

or: Why we reverse the Solenoids

In the absence of external fields and energy loss in materials, the angular momentum of a particle is conserved.

But a particle entering a solenoidal field will cross radial field components and its angular ($r\ p_\phi$) momentum will change.

$$\Delta(p_\phi) \; = \; \Delta\left(\frac{c\ B_z\ r}{2}\right)$$

If in the absence of the field the particle had "canonical" angular momentum $(p_\phi\ r)_{\mathrm{can}}$, then in the field it will have angular momentum:

$$p_\phi\ r \; = \; (p_\phi\ r)_{\mathrm{can}} \; + \; \left(\frac{c\ B_z\ r}{2}\right)\ r$$

so

$$(p_\phi\ r)_{\mathrm{can}} \; = \; p_\phi\ r \; - \; \left(\frac{c\ B_z\ r}{2}\right)\ r$$

If the initial canonical angular momentum is zero, then in B_z:

$$p_\phi\ r \; = \; \left(\frac{c\ B_z\ r}{2}\right)\ r$$

Material will reduce all momenta, both longitudinal and transverse.
Re-acceleration will not change the angular momenta.

The angular momentum will continuously fall.

Consider the case of almost complete transverse cooling: all transverse momenta are reduced to near zero leaving the beam streaming parallel to the axis.

$$p_\phi \; r \;\approx\; 0$$

and

$$(p_\phi \; r)_{\rm can} \;=\; p_\phi \; r \;-\; \left(\frac{c\; B_z \; r}{2}\right) \; r \;=\; -\left(\frac{c\; B_z \; r}{2}\right) \; r$$

When the beam exits the solenoid, then this canonical angular momentum becomes a real angular momentum and represents an effective emittance, and severely limits the possible cooling.

$$p_\phi \; r(end) \;=\; -\left(\frac{c\; B_z \; r}{2}\right) \; r$$

The only reasonable solution is to reverse the field, either once, a few, or many times.

1.6.1 Single Field Reversal Method

The minimum required number of field "flips" is one.

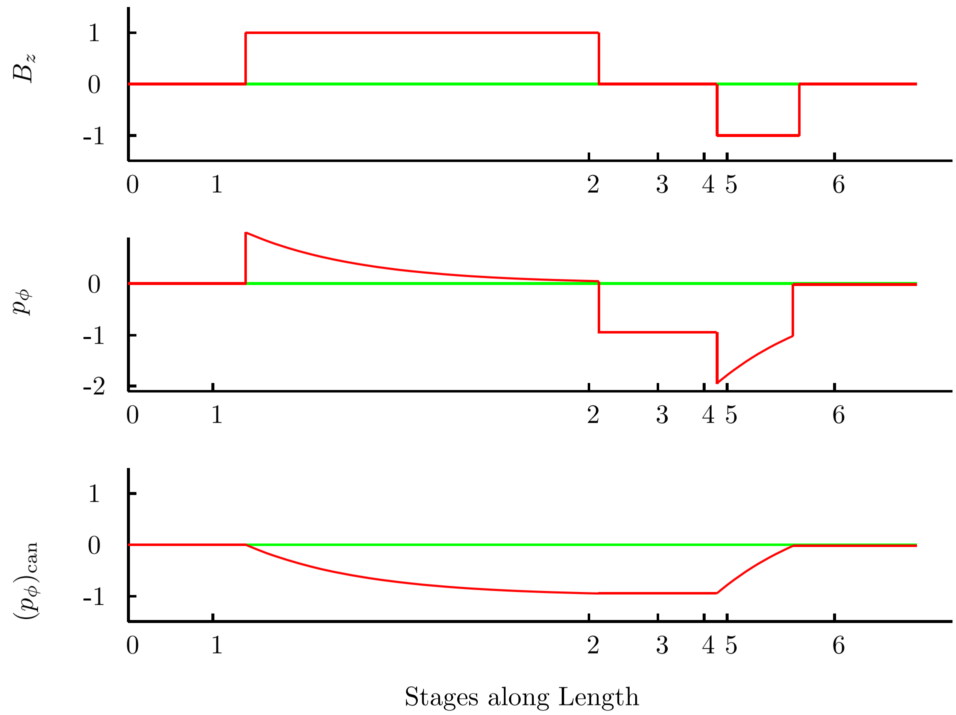


Figure: Axial Field, Angular Momentum, and Canonical Angular Momentum, in an Ideal, Single Field Reversal, Solenoid Cooling System.

After exiting the first solenoid, we have real coherent angular momentum:

$$(p_\phi \ r)_3 \ = \ -\left(\frac{c \ B_{z1} \ r}{2}\right) \ r$$

The beam now enters a solenoid with opposite field $B_{z2} = -B_{z1}$.

The canonical angular momentum remains the same, but the real angular momentum is doubled.

$$(p_\phi \ r)_4 \ = \ -2\left(\frac{c \ B_{z1} \ r}{2}\right) \ r$$

We now introduce enough material to halve the transverse field components. Then

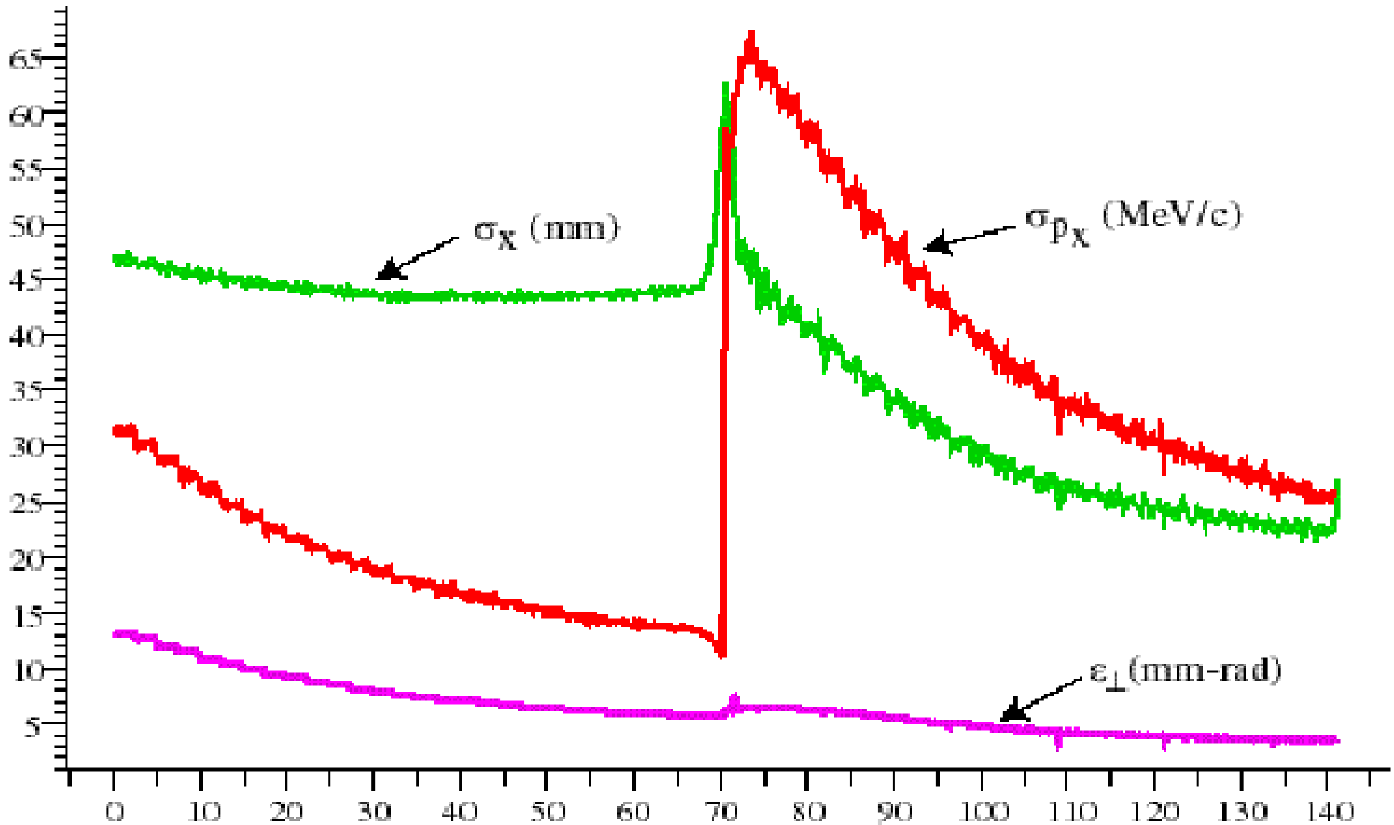
$$(p_\phi \ r)_5 \ = \ -\left(\frac{c \ B_{z1} \ r}{2}\right) \ r$$

This is inside the field $B_{z2} = -B_{z1}$. The canonical momentum, and thus the angular momentum on exiting, is now:

$$(p_\phi \ r)_6 \ = \ -\left(\frac{c \ B_{z1} \ r}{2}\right) \ r \ - \ -\left(\frac{c \ B_{z1} \ r}{2}\right) \ r \ = \ 0$$

1.6.2 Example of "Single Flip"

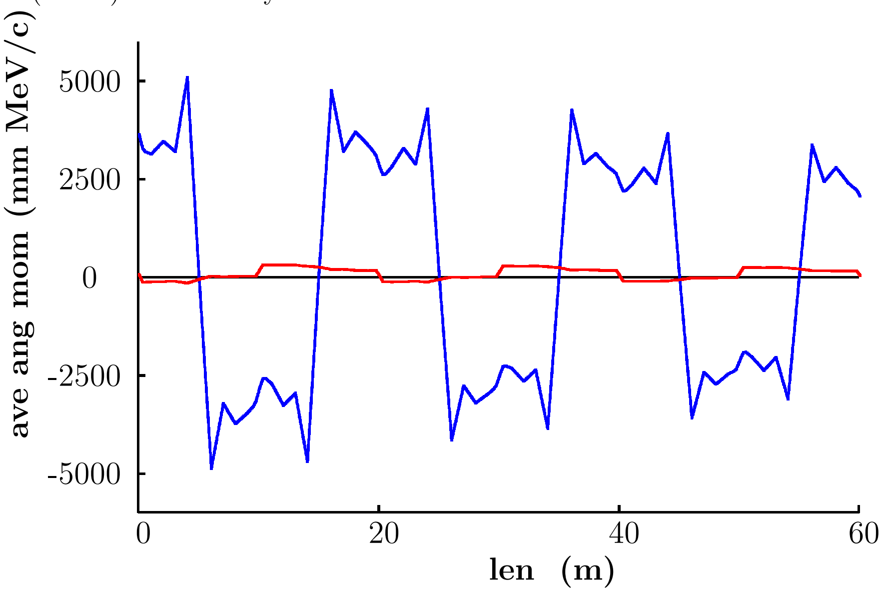
From "single flip alternative" in US Study 2



1.6.3 Alternating Solenoid Method

If we reverse the field frequently enough, no significant canonical angular momentum is developed.

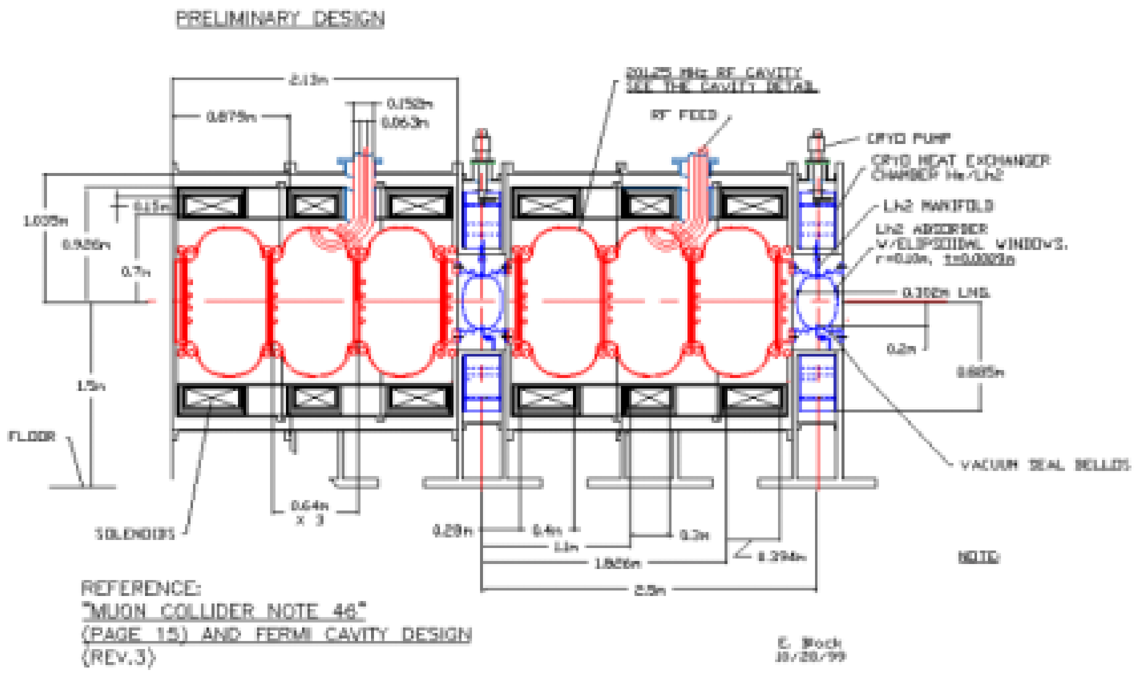
The Figure below shows the angular momenta and canonical angular momenta in a simulation of an "alternating solenoid" cooling lattice. It is seen that while the coherent angular momenta are large, the canonical angular momentum (in red) remains very small.



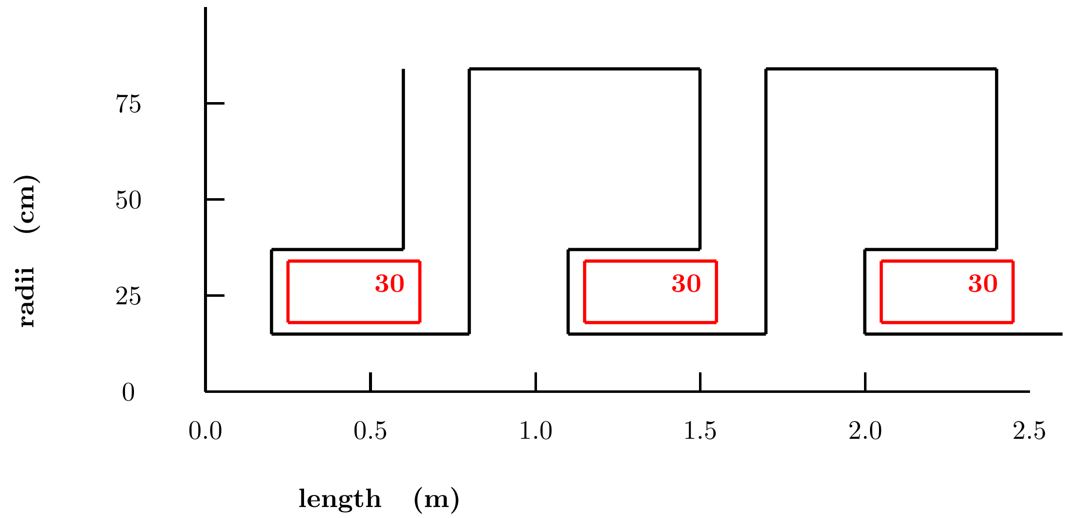
1.7 Focussing Lattice Design

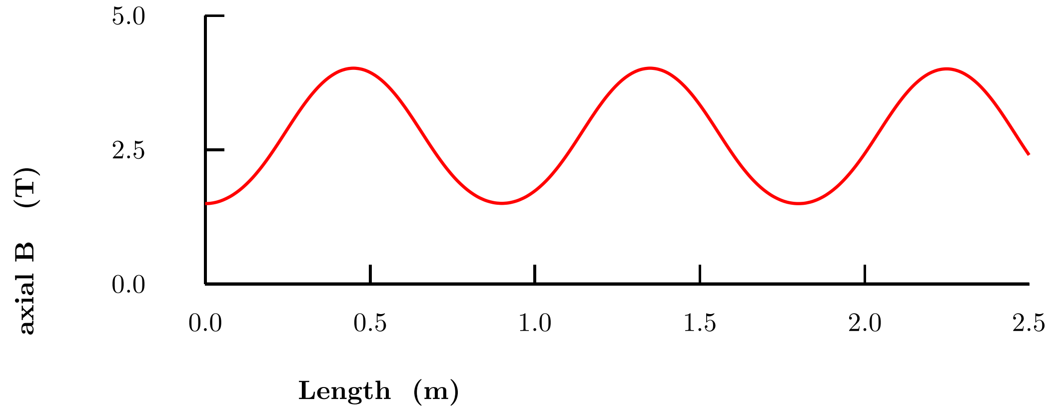
1.7.1 Solenoids with few "flips"

- Coils Outside RF: e.g. FNAL 1 flip



- Coils interleeved: e.g. CERN



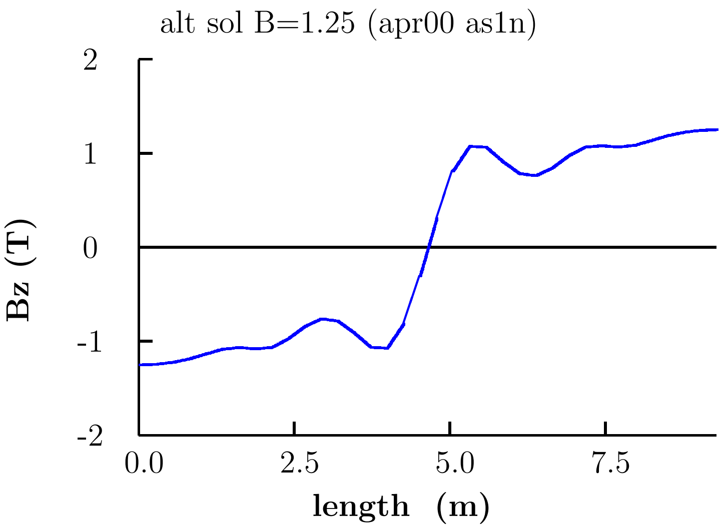


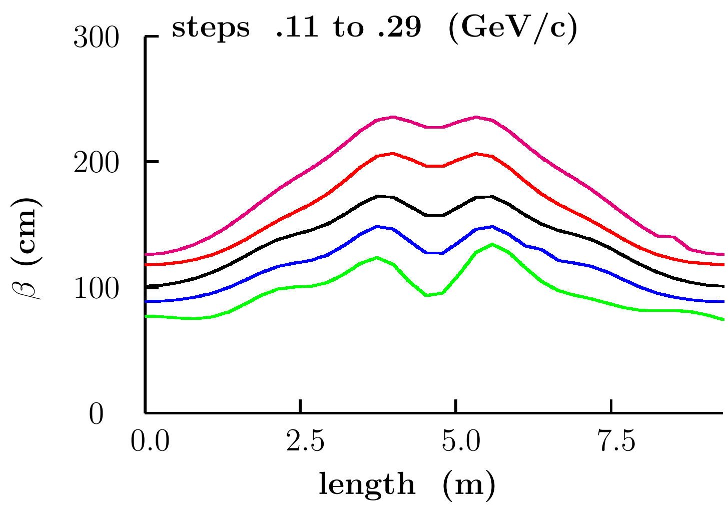
Note: Field is far from uniform and must be treated as a lattice.

"Flips"

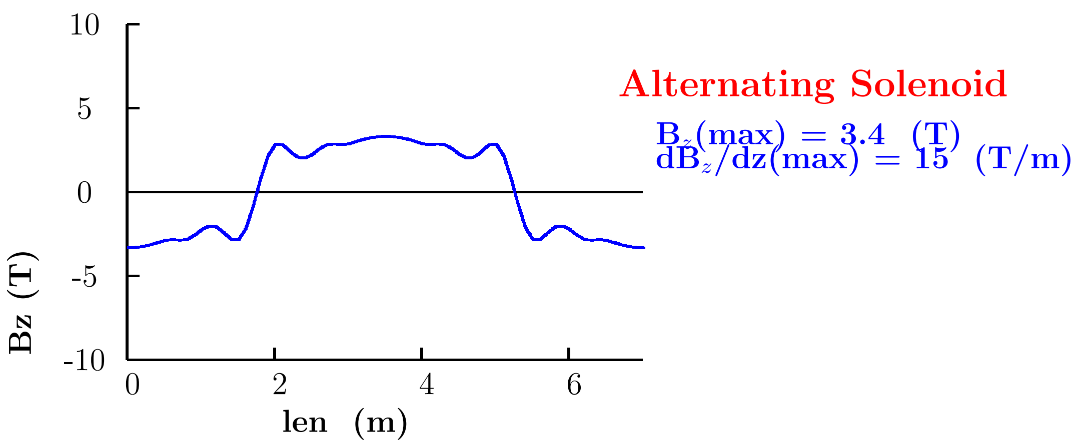
One must design the flips to match the betas from one side to the other.

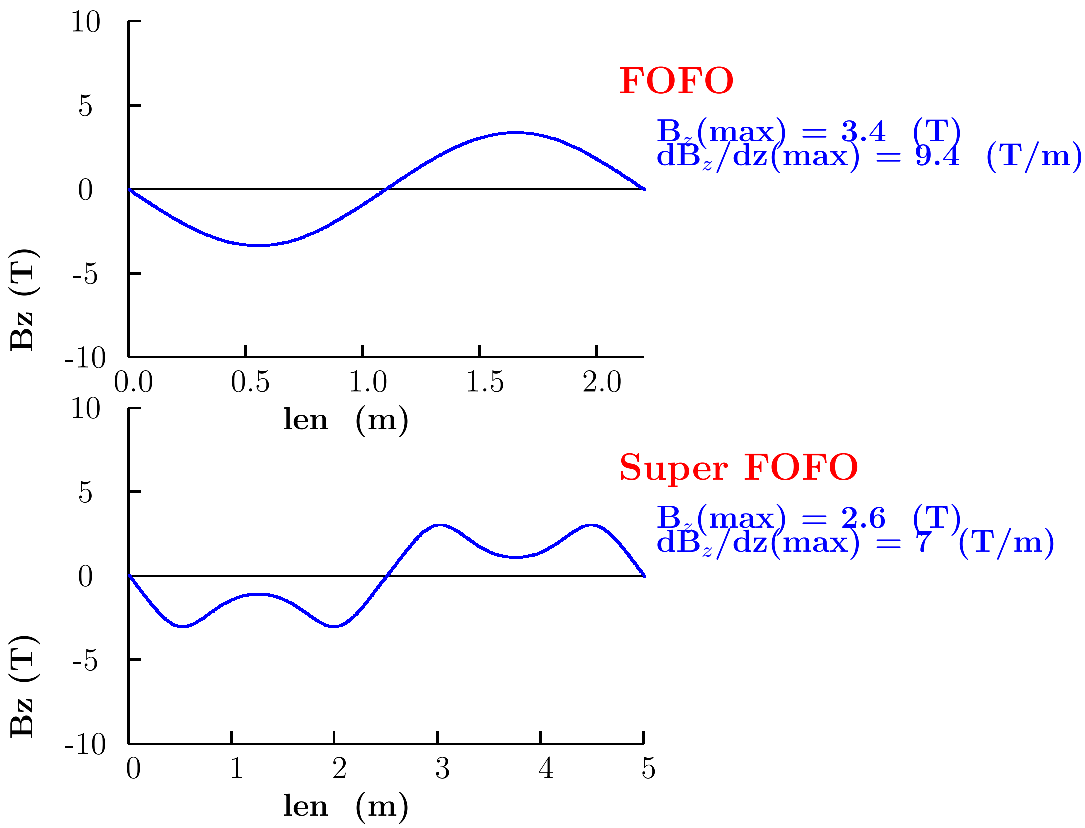
For a computer matched flip, the following figure shows B_z vs. z and the $\beta_\perp$'s vs. z for different momenta.





1.7.2 Lattices with many "flips"

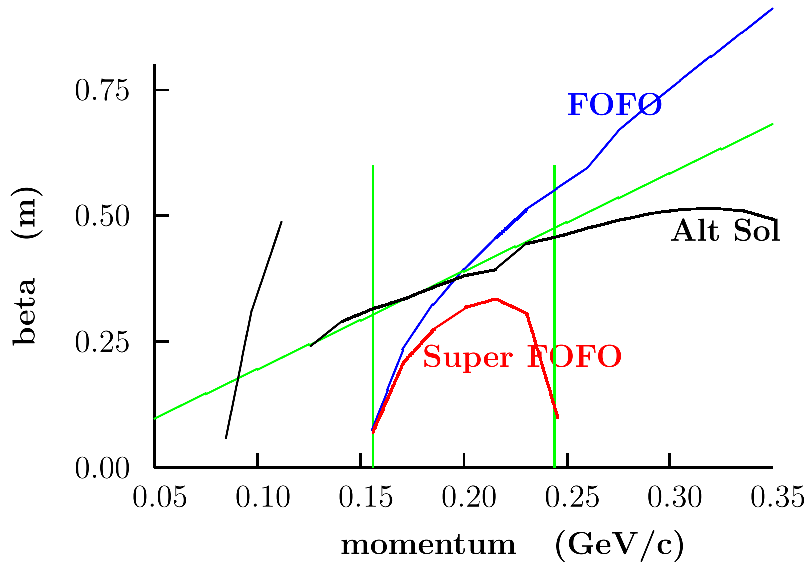




Determination of lattice betas

- Track single near paraxial particle through many cells
- plot θ_x vs x after each cell
- fit ellipse: $\beta_{x,y}$ = A((x) / A(θ_x)

beta vs. Momentum

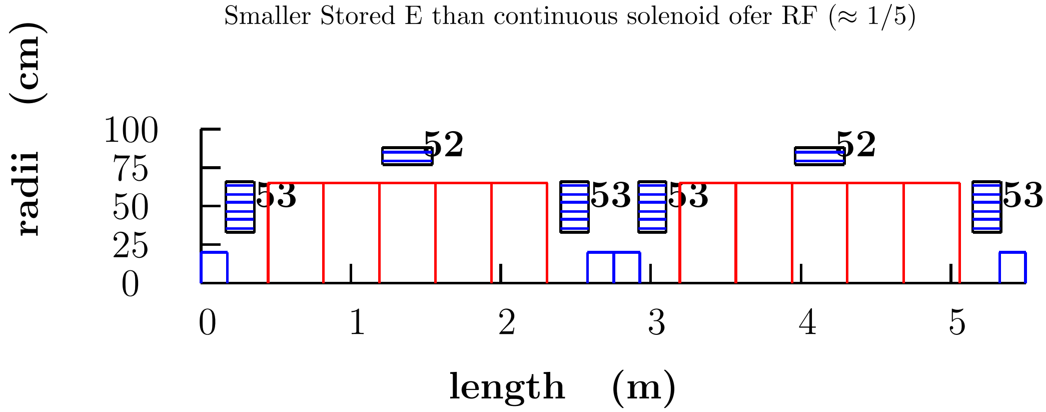


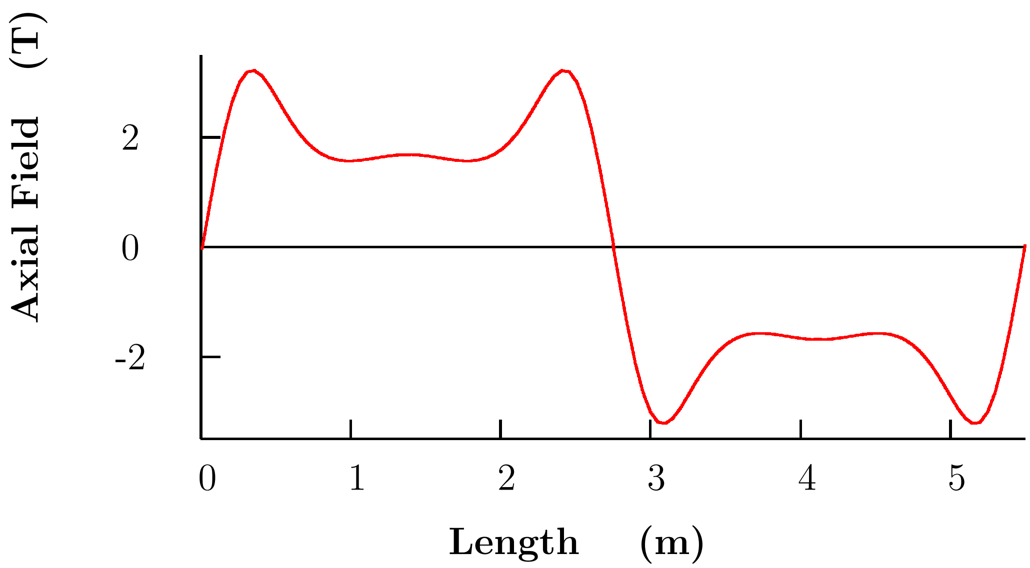
- Solenoid has largest p acceptance
- FOFO shows $\beta \ \propto \ dp/p$
- SFOFO more complicated, and better

1.7.3 Example of Multi-flip lattice

US Study 2 Super FOFO

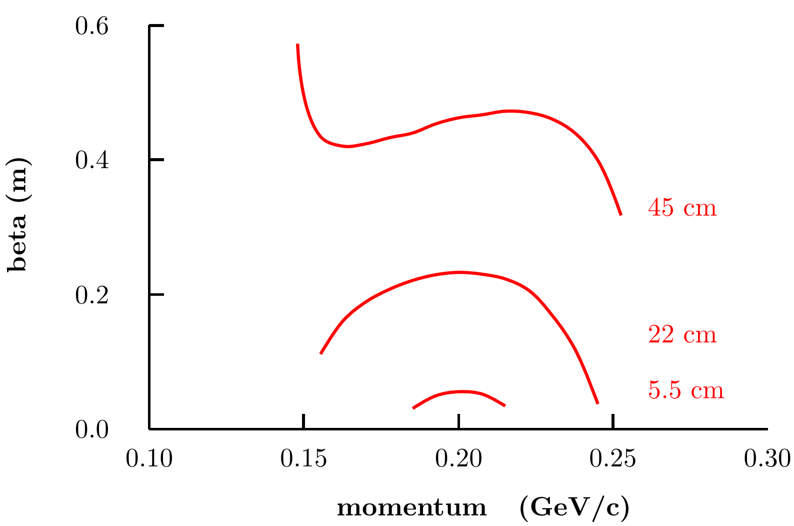
Smaller Stored E than continuous solenoid ofer RF ($\approx 1/5$)





Adjusting Currents adjusts $\beta_\perp$'s

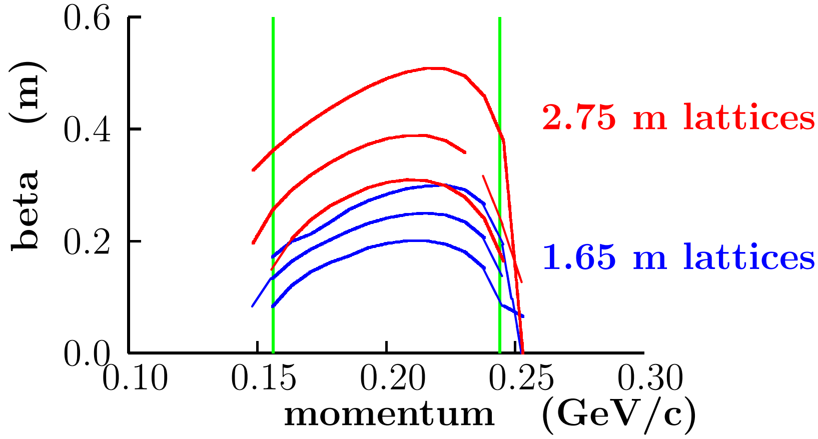
But mom acceptance falls with $\beta_\perp$

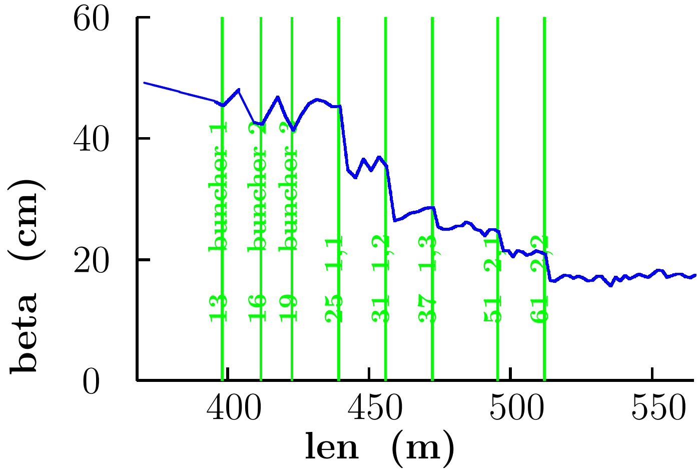


This allows:

1.7.4 Tapering the Cooling Lattice

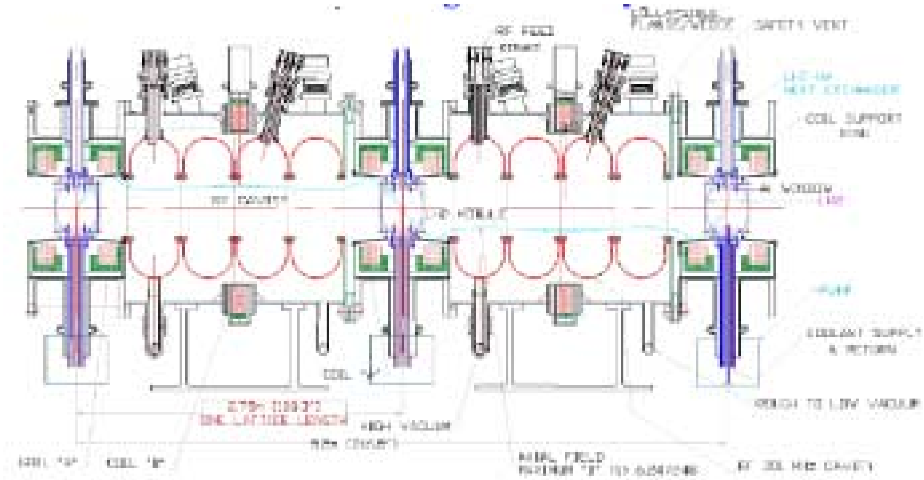
- **as emittance falls, lower betas**
- **maintain constant angular beam size**
- **maximizes cooling rate**
- **Adjust current, then lattice**

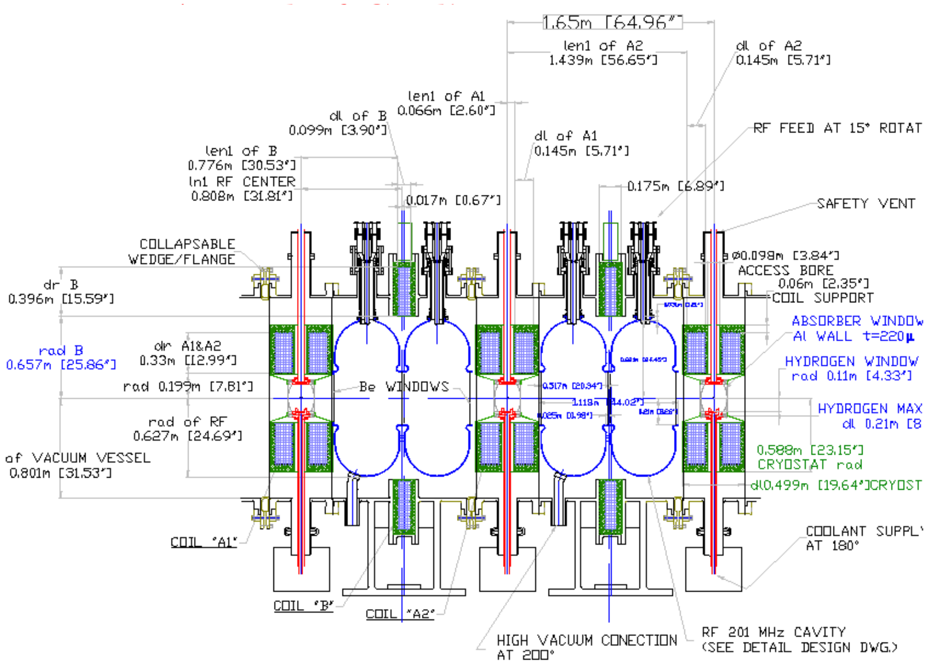




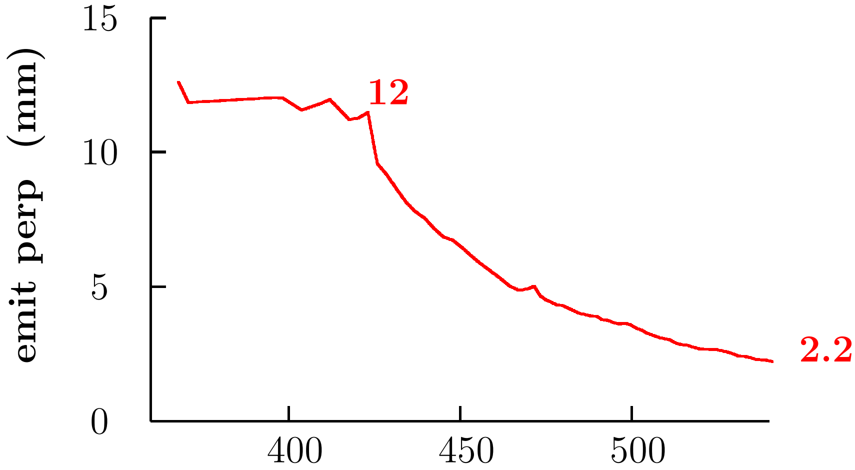
1.7.5 Hardware

At Start of Cooling





1.7.6 Study 2 Performance



With RF and Hydrogen Windows, $C_o \approx 45\ 10^{-4}$
$\beta_\perp$(end)=.18 m, β_v(end)=0.85, So

$$\epsilon_\perp(\mathrm{min}) = \frac{45\ 10^{-4}\ 0.18}{0.85} = 0.95\ \ (\pi mm\ mrad)$$

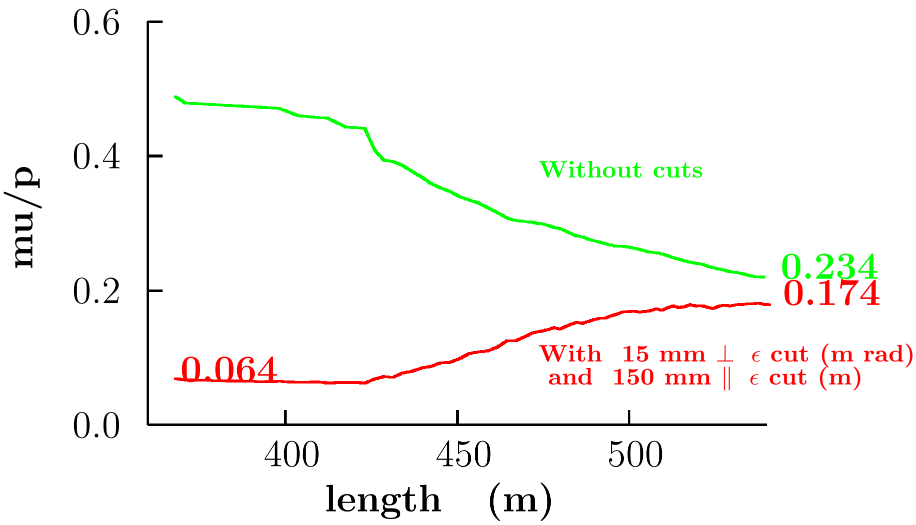
$$\frac{\epsilon_\perp}{\epsilon_\perp(\mathrm{min})} \approx 2.3$$

so from eq. 7

$$\frac{d\epsilon}{\epsilon}(\mathrm{end}) = \left(1 - \frac{\epsilon}{\epsilon(\mathrm{min})}\right)\frac{dp}{p} \approx 0.57\,\frac{dp}{p}$$

- A lower emittance would req. >> length

Muons accepted by Acceleration



- Gain Factor = 3
- No Further gain from length
- Loss from growth of long emit.
- Avoided if longitudinal cooling