NUFAC 04 Institute Lectures

Tokyo July 16-24, 2004

R. B. Palmer (BNL)

- **Solenoid Focus & Transverse Ionization Cooling**
- **6 Dimension Ionization Cooling Rings**

http://pubweb.bnl.gov/people/palmer/04school/04school.pdf & .ps
http://pubweb.bnl.gov/people/palmer/04school/04tutorial.pdf & .ps
http://pubweb.bnl.gov/people/palmer/04school/icoolman.pdf & .ps
http://pubweb.bnl.gov/people/fernow/icool/v277/ & .ps

1 Preface

1.1 Units

When discussing the motion of particles in magnetic fields, I will use MKS units, but this means that momentum, energy, and mass are in Joules and kilograms, rather than in the familiar 'electron Volts'. To make the conversion easy, I will introduce these quantities in the forms: $[pc/e]$, $[E/e]$, and $[mc^2/e]$, respectively. Each of these expressions are then in units of straight Volts corresponding to the values of p, E and m expressed in electron Volts. For instance, I will write, for the bending radius in a field B:

$$\rho = \frac{[pc/e]}{B\ c}$$

meaning that the radius for a 3 GeV/c particle in 5 Tesla is

$$\rho = \frac{3\ 10^9}{5 \times 3\ 10^8} = 2m$$

This units problem is often resolved in accelerator texts by expressing parameters in terms of $(B\rho)$ where this is a measure of momentum: the momentum that would have this value of $B \times \rho$, where

$$(B\rho) = \frac{[pc/e]}{c}$$

For 3 GeV/c, $(B\rho)$ is thus 10 (Tm), and the radius of bending in a field B=5 (T) is:

$$\rho = \frac{(B\rho)}{B} = \frac{10}{5} = 2m$$

1.2 Useful Relativistic Relations

$$dE = \beta_v \, dp \tag{1}$$

$$\frac{dE}{E} = \beta_v^2 \, \frac{dp}{p} \tag{2}$$

$$d\beta_v = \frac{dp}{\gamma^2} \tag{3}$$

Note: I use β_v to denote v/c to distinguish it from the Courant-Schneider $\beta_\perp$

1.3 Emittance

$$\text{normalized emittance} = \frac{\text{Phase Space Area}}{\pi \text{ m c}}$$

The phase space can be transverse:p_x vs x, p_y vs y, or longitudinal Δp_z vs z, where Δp_z and z are with respect to the moving bunch center.

If x and p_x are both Gaussian and uncorrelated, then the area is that of an upright ellipse, and:

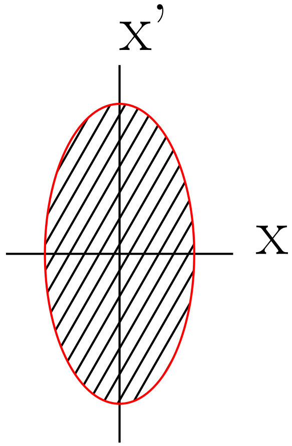
$$\epsilon_\perp = \frac{\pi \ \sigma_{p_\perp} \sigma_x}{\pi \ mc} = (\gamma\beta_v)\sigma_\theta\sigma_x \qquad (\pi \ m \ rad) \tag{4}$$

$$\epsilon_\parallel = \frac{\pi \ \sigma_{p_\parallel} \sigma_z}{\pi \ mc} = (\gamma\beta_v)\frac{\sigma_p}{p} \ \sigma_z \qquad (\pi \ m \ rad) \tag{5}$$

$$\epsilon_6 = \epsilon_\perp^2 \ \ \epsilon_\parallel \qquad (\pi \ m)^3 \tag{6}$$

Note that the π, added to the dimension, is a reminder that the emittance is phase space/π

1.4 Beta$_{Courant-Schneider}$ of Beam



Upright phase ellipse in x' vs x,

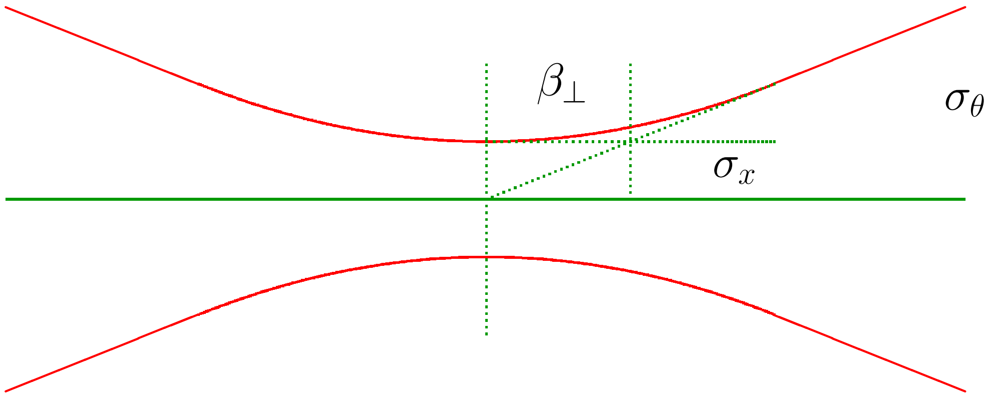
$$\beta_\perp = \left(\frac{\text{width}}{\text{height}} \quad \text{of phase ellipse} \right) = \frac{\sigma_x}{\sigma_\theta} \tag{7}$$

Then, using emittance definition:

$$\sigma_x = \sqrt{\epsilon_\perp \, \beta_\perp \, \frac{1}{\beta_v \gamma}} \tag{8}$$

$$\sigma_\theta = \sqrt{\frac{\epsilon_\perp}{\beta_\perp} \, \frac{1}{\beta_v \gamma}} \tag{9}$$

1.5 Beta$_{Courant-Schneider}$ at focus



$$\sigma_x = \sigma_o \sqrt{1 + \left(\frac{z}{\beta_\perp}\right)^2}$$

$\beta_\perp$ is like a depth of focus

As $z \rightarrow \infty$

$$\sigma_x \rightarrow \frac{\sigma_o\ z}{\beta_\perp}$$

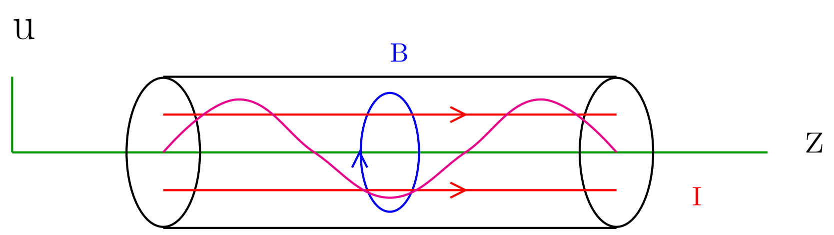
giving an angular spread of

$$\theta = \frac{\sigma_o}{\beta_\perp}$$

as above in eq.7

1.6 Beta$_{Courant-Schneider}$ of a Lattice

$\beta_\perp$ above was defined by the beam, but a lattice can have a β_o that "matches" a beam e,g. if continuous inward focusing force, as in a current carrying lithium cylinder (lithium lens), then



$$\frac{d^2u}{dz^2} = -k\ u$$

$$u = A\ \sin\left(\frac{z}{\beta_o}\right) \qquad u' = \frac{A}{\beta_o}\ \cos\left(\frac{z}{\beta_o}\right)$$

where $\beta_o = 1/\sqrt{k}$

$$\frac{\text{width}}{\text{height}} \text{ of phase ellipse } = \frac{\hat{u}}{\hat{u'}} = \beta_o$$

If $\beta_o = \beta_{\text{beam}}$ then all particles move arround the ellipse, and the shape, and thus β_{beam} remains constant. i.e. the beam is matched to this lattice. If $\beta_o \neq \beta_{\text{beam}}$, then β_{beam} oscillates about β_o: often refered to as a "beta beat".

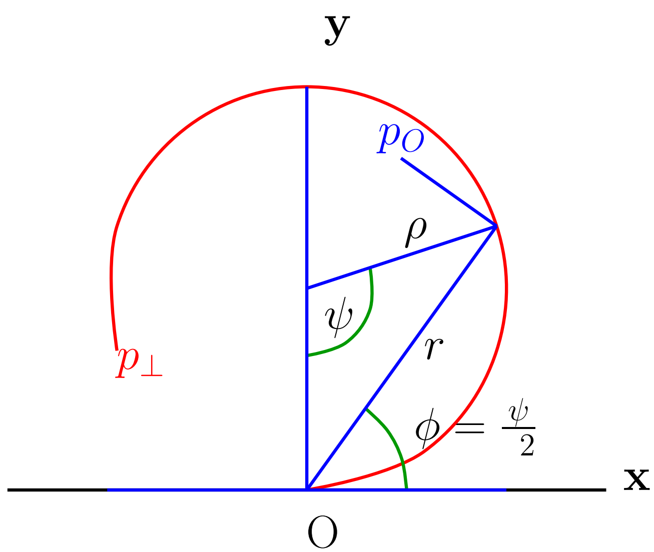
Note:

$$\lambda = 2\pi\ \beta_o \tag{10}$$

1.7 Introduction to Solenoid Focussing

1.7.1 Motion in Long Solenoid

Consider motion in a fixed axial filed B_z, starting on the axis **O** with finate transverse momentum $p_\perp$ **i.e. with initial angular momentum=0.**



$$\rho = \frac{[pc/e]_\perp}{c\,B_z} \tag{11}$$

$$x = \rho \sin(\psi)$$

$$y = \rho\,(1 - \cos(\psi))$$

$$\frac{dz}{d\psi} = \rho\,\frac{p_z}{p_\perp}$$

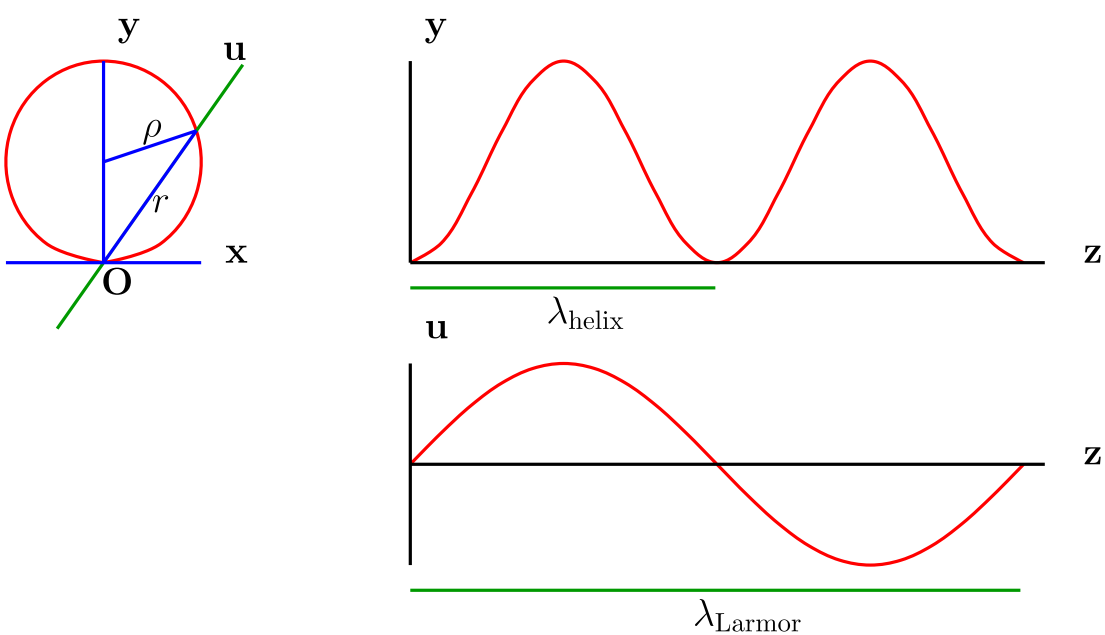
For $\psi < 180^o \quad \phi < 90^o$:

$$r = 2\rho\ \sin\left(\frac{\psi}{2}\right) = 2\rho \sin(\phi)$$

$$\frac{dz}{d\phi} = 2\,\rho\,\frac{p_z}{p_\perp}$$

1.7.2 Larmor Plane

If The center of the solenoid magnet is at **O**, then consider a plane that contains this axis and the particle. This, for a particle with initally no angular momentum, is the 'Larmor Plane:



$$u = 2\rho \ \sin(\phi) \tag{12}$$

$$\lambda_{\text{Helix}} = 2\pi \frac{dz}{d\psi} = 2\pi \ \rho \frac{p_z}{p_\perp} = 2\pi \frac{[pc/e]_z}{c \ B_z}$$

$$\lambda_{\text{Larmor}} = 2\pi \frac{dz}{d\phi} = 2\pi \ 2\rho \frac{p_z}{p_\perp} = 4\pi \frac{[pc/e]_z}{c \ B_z}$$

The lattice parameter β_o is defined in the Larmor frame, so

$$\beta_o = \frac{\lambda_{Larmor}}{2\pi} = \frac{2 \ [pc/e]_z}{c \ B_z} \tag{13}$$

1.7.3 Focusing Force

In this constant B case, the observed sinusoidal motion in the u plane is generated by a restoring force towards the axis **O**.

The momentum p_O about the axis **O** (perpendicular to the Larmor plane), using eq.11 and eq.12:

$$[p_O c/e] \;=\; [p_\perp c/e]\;\sin(\phi) \;=\; cB_z\rho\,\frac{u}{2\rho} \;=\; \frac{cB_z}{2}\;u \tag{14}$$

And the inward bending as this momentum crosses the B_z field is

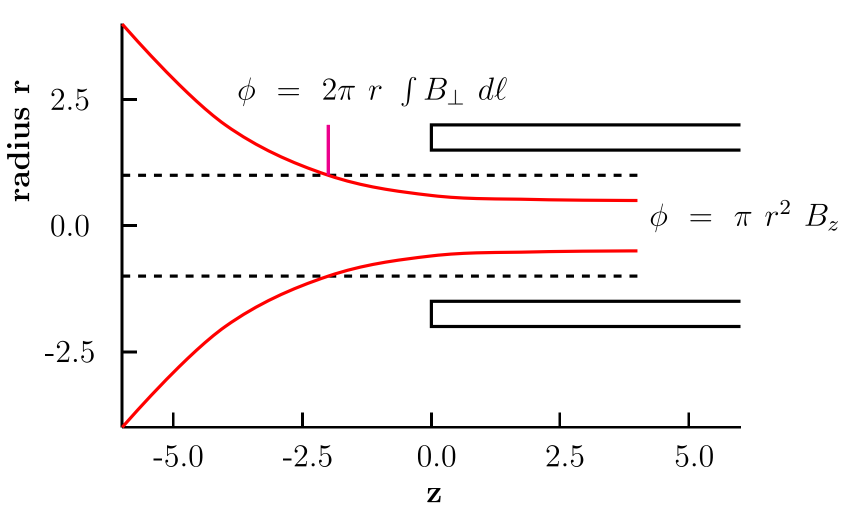
$$\frac{d^2u}{dz^2} \;=\; -\left(\frac{cB_z}{2\;[p_z c/e]}\right)^2 \;u \tag{15}$$

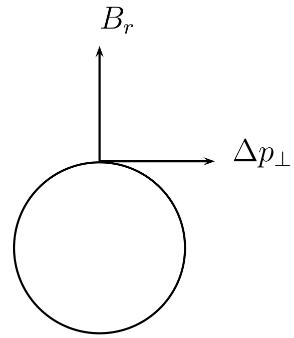
This inward force proportional to the distance u from the axis **is an ideal focusing force**

Note: the focusing "Force" $\propto\; B_z^2$ so it works the same for either sign, and $\propto\; 1/p_z^2$. Wheras in a quadrupole the force $\propto\; 1/p$ So solenoids are not good for high p, but beat quads at low p.

1.7.4 Entering a solenoid from outside

We will now look at a simple non-uniform B_z case. Let a particle start from the axis with finite transverse momentum, but **no angular momentum**. After some distance with no field, it reaches a radius u and then enters a solenoid with B_z. As it eneters the solenoid it crosses radial field lines and receives some angular momentum.





$$\Delta[pc/e]_\perp \;=\; \int B_r\; dz \;=\; \frac{B_z\; r\; c}{2} \tag{16}$$

Sof for our case with zero initial transverse momentum,

$$[pc/e]_\perp \;=\; \int B_r\; dz \;=\; \frac{B_z\; r\; c}{2}$$

Which is the same as eq.14, and will lead to the same inward bending (eq.15), as when the particle started inside the field.

In fact eq.15 is true no matter how the axial field varies

1.7.5 Canonical Angular momentum

In general, for axial symmetry, a particle will have a conserved "Canonical Angular Momentum" $\mathcal{M}_o$ equal to the angular momentum outside the axial fields.

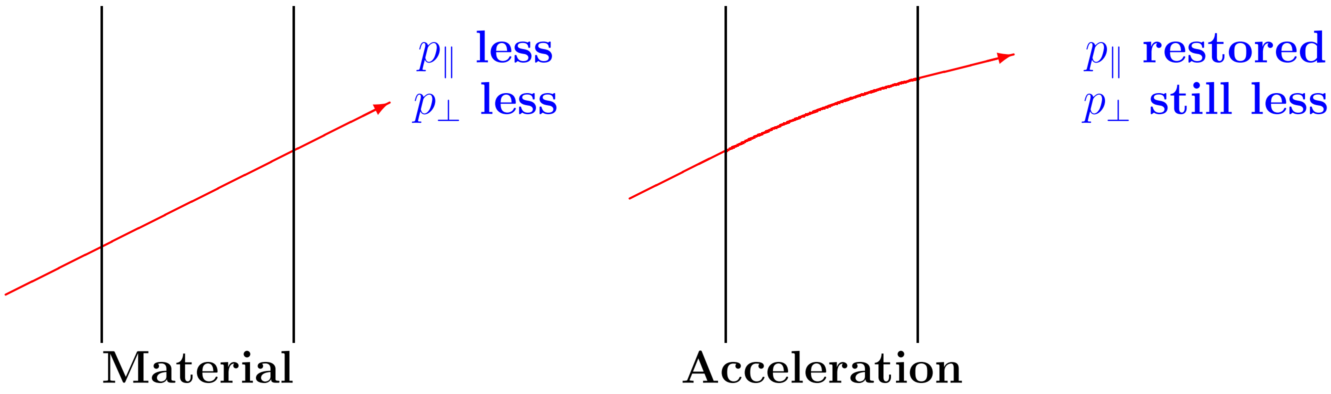
$$[\mathcal{M}c^2/e]_o \; = \; p_\perp \, r \text{ (Outdise the field)}$$

Inside a varying field $B_z(z)$, the real angular momentum will be:

$$[\mathcal{M}c^2/e] \; = \; [\mathcal{M}c^2/e]_o \; + \; \frac{r^2 \, B_z \, c}{2}$$

But in the rotating Larmor Frame the anular momentum is always just the Canonical angular momentum, and motion in that frame has only inward focusing forces, with no angular kicks.

2 Transverse Cooling



2.0.6 Cooling rate vs. Energy

$$\text{(eq 4)} \quad \epsilon_{x,y} = \gamma\beta_v\, \sigma_\theta\, \sigma_{x,y}$$

If there is no Coulomb scattering, or other sources of emittance heating, then σ_θ and $\sigma_{x,y}$ are unchanged by energy loss, but p and thus $\beta\gamma$ are reduced. So the fractional cooling $d\epsilon\ /\epsilon$ is (using eq.2):

$$\frac{d\epsilon}{\epsilon} = \frac{dp}{p} = \frac{dE}{E}\,\frac{1}{\beta_v^2} \tag{17}$$

which, for a given energy change, strongly favors cooling at low energy.

But if total acceleration were not important, e.g. if the cooling is done in a ring, then there is another criterion: The cooling per fractional loss of particles by decay:

$$Q = \frac{d\epsilon/\epsilon}{dn/n} = \frac{dp/p}{d\ell/c\beta_v\gamma\tau}$$

$$= \frac{dE/E \;\; 1/\beta_v^2}{d\ell/(c\gamma\beta_v\tau)}$$

$$= (c\tau/m_\mu) \;\; \frac{dE}{d\ell} \;\; \frac{1}{\beta_v}$$

Which only mildly favours low energy

2.0.7 Heating Terms

$$\epsilon_{x,y} = \gamma\beta_v \; \sigma_\theta \; \sigma_{x,y}$$

Between scatters the drift conserves emittance (Liouiville).

When there is scattering, $\sigma_{x,y}$ is conserved, but σ_θ is increased.

$$\Delta(\epsilon_{x,y})^2 = \gamma^2\beta_v^2 \; \sigma_{x,y}^2 \Delta(\sigma_\theta^2)$$

$$2\epsilon \; \Delta\epsilon = \gamma^2\beta_v^2 \left(\frac{\epsilon\beta_\perp}{\gamma\beta_v}\right) \; \Delta(\sigma_\theta^2)$$

$$\Delta\epsilon = \frac{\beta_\perp\gamma\beta_v}{2} \; \Delta(\sigma_\theta^2)$$

e.g. from Particle data booklet

$$\Delta(\sigma_\theta^2) \approx \left(\frac{14.1\ 10^6}{[pc/e]\beta_v}\right)^2 \ \frac{\Delta s}{L_R}$$

$$\Delta\epsilon \ = \ \frac{\beta_\perp}{\gamma\beta_v^3}\,\Delta E \ \ \left(\left(\frac{14.1\ 10^6}{2[mc^2/e]_\mu}\right)^2 \ \frac{1}{L_R dE/ds}\right)$$

Defining

$$C(mat, E) \ = \ \frac{1}{2}\ \left(\frac{14.1\ 10^6}{[mc^2/e]_\mu)}\right)^2 \ \frac{1}{L_R\ d\gamma/ds} \tag{18}$$

then

$$\frac{\Delta\epsilon}{\epsilon} \ = \ dE\ \ \frac{\beta_\perp}{\epsilon\gamma\beta_v^3}\ \ C(mat, E) \tag{19}$$

Equating this with equation 17

$$dE\ \frac{1}{\beta_v^2\ E} \ = \ dE\ \ \frac{\beta_\perp}{\epsilon\gamma\beta_v^3}\ \ C(mat, E)$$

gives the equilibrium emittance ϵ_o:

$$\epsilon_{x,y}(min) \ = \ \frac{\beta_\perp}{\beta_v}\ \ C(mat, E) \tag{20}$$

At energies such as to give minimum ionization loss, the constant C_o for various materials are approximately:

material	T ^{o}K	density kg/m^3	dE/dx MeV/m	L_R m	C_o 10^{-4}
Liquid H_2	20	71	28.7	8.65	38
Liquid He	4	125	24.2	7.55	51
LiH	300	820	159	0.971	61
Li	300	530	87.5	1.55	69
Be	300	1850	295	0.353	89
Al	300	2700	436	0.089	248

Clearly Liquid Hydrogen is far the best material, but has cryogenic and safety complications, and requires windows made of Aluminum or other material which will significantly degrade the performance.

2.0.8 Rate of Cooling

$$\frac{d\epsilon}{\epsilon} = \left(1 - \frac{\epsilon_{\rm min}}{\epsilon}\right) \frac{dp}{p} \tag{21}$$

2.0.9 Beam Divergence Angles

$$\sigma_\theta \ = \ \sqrt{\frac{\epsilon_\perp}{\beta_\perp\ \beta_v \gamma}}$$

so, from equation 20, for a beam in equilibrium

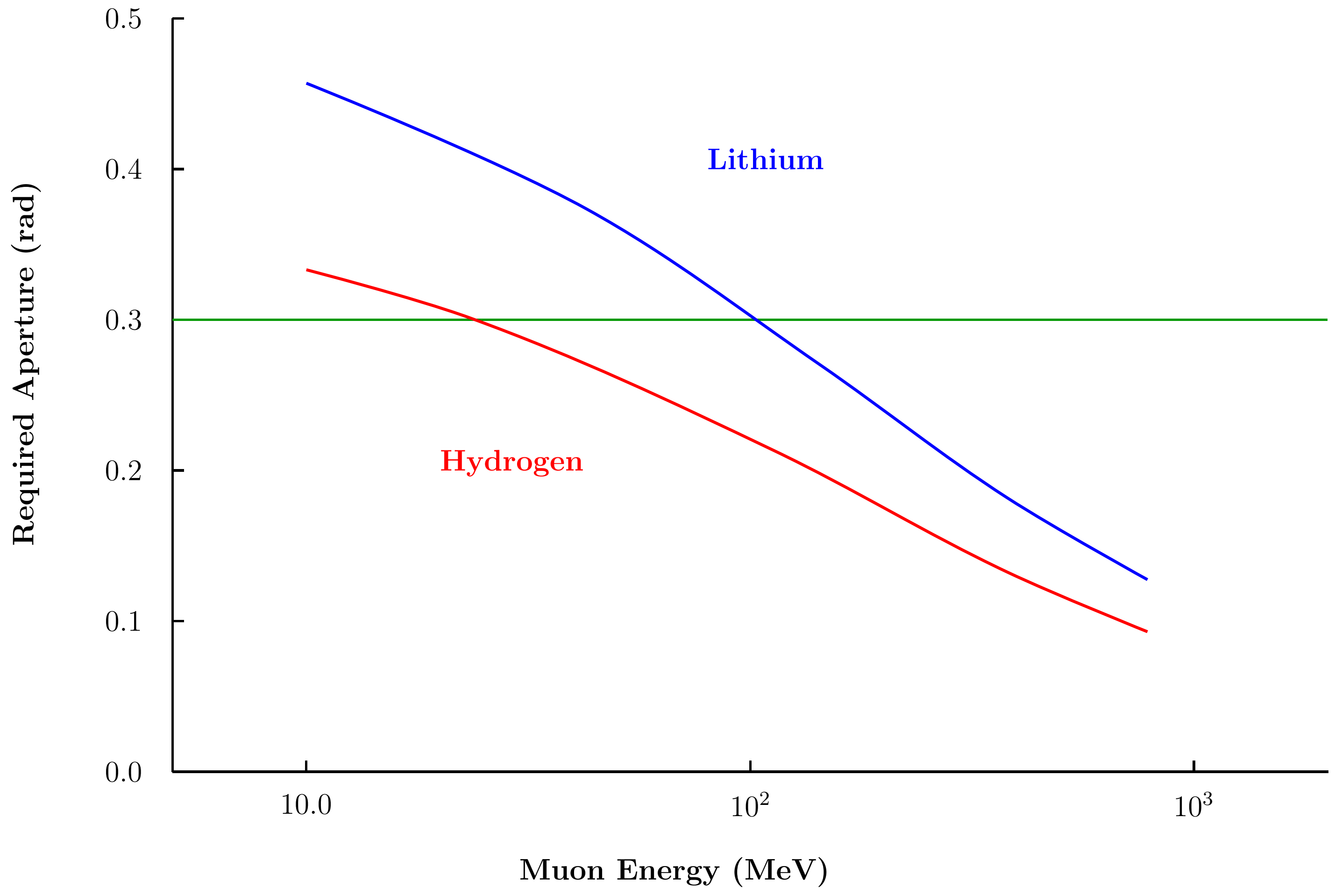
$$\sigma_\theta \ = \ \sqrt{\frac{C(mat, E)}{\beta_v^2 \gamma}}$$

and for 50 % of maximum cooling rate and an aperture at 3 σ, the angular aperture $\mathcal{A}$ of the system must be

$$\mathcal{A} \ = \ 3\sqrt{2}\ \sqrt{\frac{C(mat, E)}{\beta_v^2 \gamma}} \tag{22}$$

Apertures for hydrogen and lithium are plotted vs. energy below. These are **very large angles**, and if we limit apertures to less than 0.3, then this requirement sets lower energy limits of about 100 MeV ($\approx$ 170 MeV/c) for Lithium, and about 25 MeV ($\approx$ 75 MeV/c) for hydrogen.

In fact $\theta = 0.3$ is optimistic, as we will see in the tutorial.



0.5
0.4
0.3
0.2
0.1
0.0
Required Aperture (rad)
Lithium
Hydrogen
10.0
10^2
10^3
Muon Energy (MeV)

2.1 Focusing Systems

2.1.1 Solenoid

In a solenoid with axial field B_{sol} (from eq 13)

$$\beta_\perp = \frac{2\ [pc/e]}{c\ B_{sol}}$$

so

$$\epsilon_{x,y}(min) = C(mat, E)\ \frac{2\ \gamma\ [mc^2/e]_\mu}{B_{sol}\ c} \tag{23}$$

For $E = 100\ MeV$ $(p \approx 170\ MeV/c)$, $B = 20\ T$, then $\beta \approx 5.7\ cm.$ and

$\epsilon_{x,y} \approx 266(\pi mm\ mrad)$.

2.1.2 Current Carrying Rod

In a rod carrying a uniform axial current, the azimuthal magnetic field B varies linearly with the radius r. A muon traveling down it is focused:

$$\frac{d^2r}{dr^2} = -\frac{B\ c}{[pc/e]} = -\left(\frac{c}{[pc/e]}\ \frac{dB}{dr}\right)\ r$$

so orbits oscillate with

$$\beta_\perp^2 = \frac{\gamma\beta_v}{dB/dr}\ \frac{[mc^2/e]_\mu}{c} \tag{24}$$

If we set the rod radius a to be f_{ap} times the rms beam size $\sigma_{x,y}$ (from eq.8),

$$\sigma_{x,y} = \sqrt{\frac{\epsilon_{x,y}\ \beta_\perp}{\beta_v \gamma}}$$

and if the field at the surface is B_{max}, then

$$\beta_\perp^2 = \frac{\gamma\beta_v[mc^2/e]_\mu\ f_{ap}}{B_{max}\ c}\sqrt{\frac{\epsilon_{x,y}\ \beta}{\gamma\ \beta_v}}$$

from which we get:

$$\beta_\perp = \left(\frac{f_{ap}\ [mc^2/e]_\mu}{B_{max}\ c}\right)^{2/3}\ (\gamma\beta_v\ \epsilon_{x,y})^{1/3}$$

puting this in equation 20

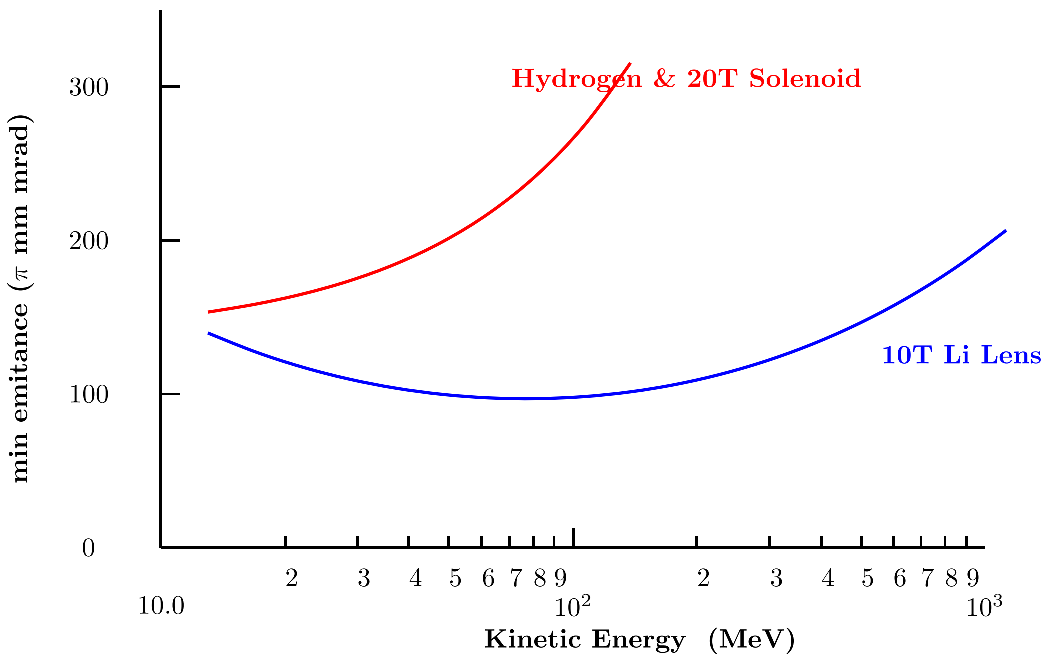
$$\epsilon_{x,y}(min) = (C(mat,E))^{1.5}\left(\frac{f_{ap}\ [mc^2/e]_\mu}{B_{max}\ c\ \beta_v}\right)\ \sqrt{\gamma} \tag{25}$$

e.g. B_{max}=10 T, f_{ap}=3, E=100 MeV, then $\beta_\perp = 1.23$ cm, and $\epsilon(min)$=100 (π mm mrad)

The choice of a maximum surface field of 10 T is set by breaking of the containing pipe in current solid Li designs. With liquid Li a higher field may be possible.

2.1.3 Compare Focusing

Comparing the methods as a function of the beam kinetic energy.



We see that, for the parameters selected, The lithium rod achieves a lower emittance than the solnoid despite its higher C value. Neither method allows transverse cooling below about 80 (π mm mrad)

A focusing lattice can, with limited momentum acceptance achieve $\beta_\perp$ less than given for a solenoid, but it probably can't beat the lithium rod.

2.2 Angular Momentum Problem

or: Why we reverse the solenoid directions

In the absence of external fields and energy loss in materials, the angular momentum of a particle is conserved.

But a particle entering a solenoidal field will cross radial field components and its angular momentum ($r\ p_\phi$) will change (eq.16).

$$\Delta([pc/e]_\phi) \;=\; \Delta\left(\frac{c\ B_z\ r}{2}\right)$$

If, in the absence of the field, the particle had "canonical" angular momentum $(p_\phi\ r)_{\mathrm{can}}$, then in the field it will have angular momentum:

$$[pc/e]_\phi\ r \;=\; (p_\phi\ r)_{\mathrm{can}} \;+\; \left(\frac{c\ B_z\ r}{2}\right)\ r$$

so

$$[pc/e]_\phi\ r)_{\mathrm{can}} \;=\; [pc/e]_\phi\ r \;-\; \left(\frac{c\ B_z\ r}{2}\right)\ r \tag{26}$$

If the initial average canonical angular momentum is zero, then in B_z:

$$< [pc/e]_\phi\ r > = \left(\frac{c\ B_z\ r}{2}\right)\ r$$

Material introduced to cool the beam, will reduce all momenta, both longitudinal and transverse, random and average.

Re-acceleration will not change the angular momenta, so the average angular momentum will continuously fall.

Consider the case of almost complete transverse cooling: all transverse momenta are reduced to near zero leaving the beam streaming parallel to the axis.

$$[pc/e]_\phi \, r \;\approx\; 0$$

and there is now a finite average canonical momentum (from eq.26):

$$< [pc/e]_\phi \, r >_{\mathrm{can}} = -\left(\frac{c\ B_z\ r}{2}\right)\ r$$

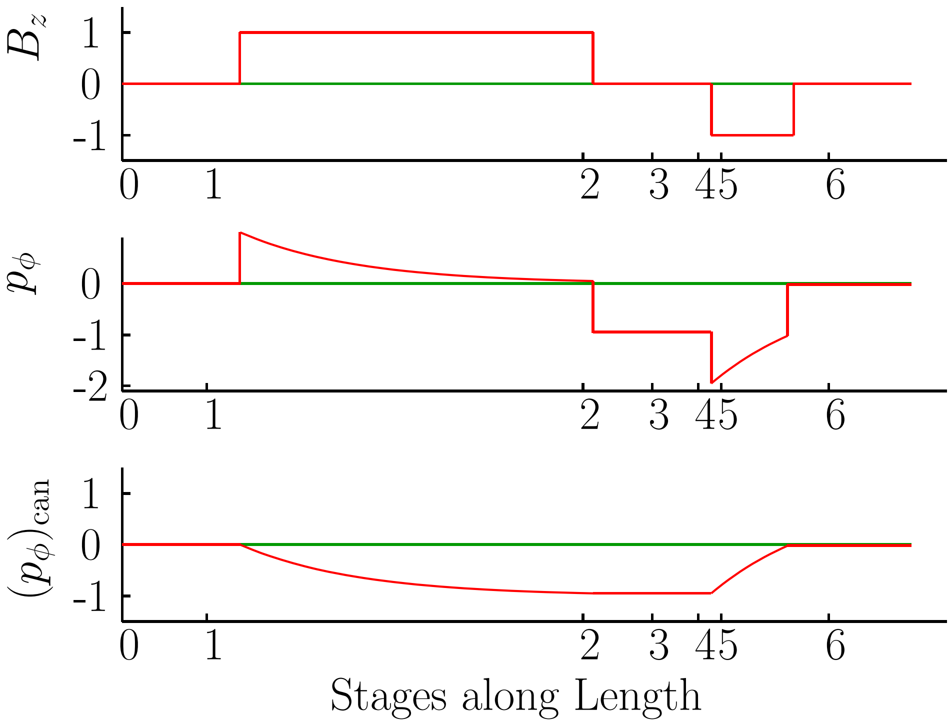
When the beam exits the solenoid, then this canonical angular momentum becomes a real angular momentum and represents an effective emittance, and severely limits the possible cooling.

$$< [pc/e]_\phi \, r >_{\mathrm{end}} = \ -\left(\frac{c\ B_z\ r}{2}\right)\ r$$

The only reasonable solution is to reverse the field, either once, a few, or many times.

2.2.1 Single Field Reversal Method

The minimum required number of field "flips" is one.



After exiting the first solenoid, we have real coherent angular momentum:

$$([pc/e]_\phi \ r)_3 \ = \ -\left(\frac{c\ B_{z1}\ r}{2}\right)\ r$$

The beam now enters a solenoid with opposite field $B_{z2} = -B_{z1}$.
The canonical angular momentum remains the same, but the real angular momentum is

doubled.

$$([pc/e]_\phi \; r)_4 \;=\; -2\left(\frac{c\; B_{z1}\; r}{2}\right)\; r$$

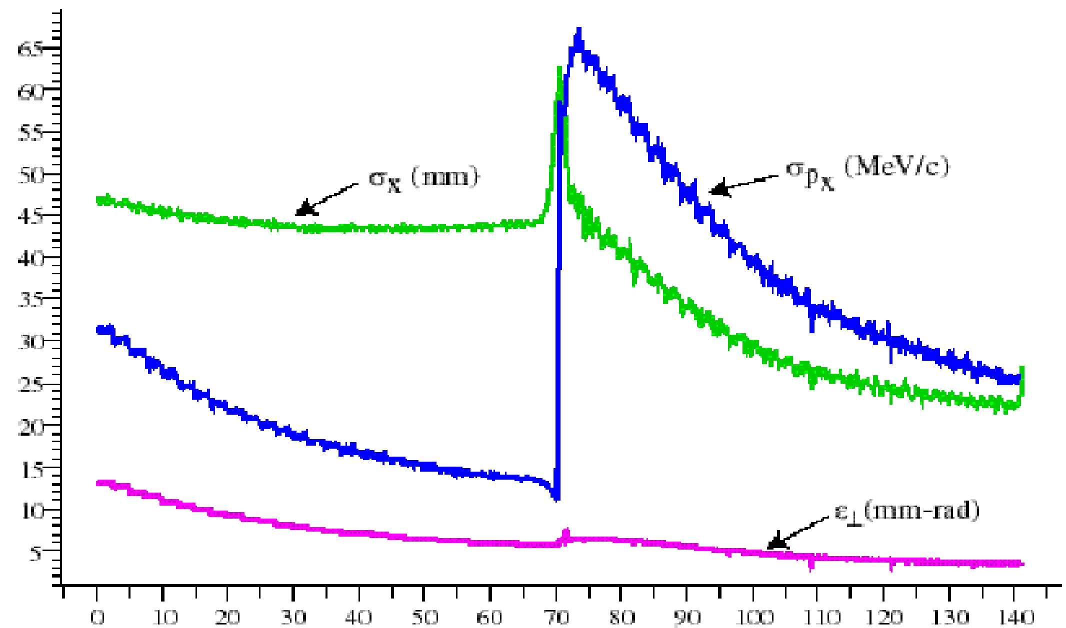
We now introduce enough material to halve the transverse field components. Then

$$([pc/e]_\phi \; r)_5 \;=\; -\left(\frac{c\; B_{z1}\; r}{2}\right)\; r$$

This is inside the field $B_{z2} \;=\; -B_{z1}$. The canonical momentum, and thus the angular momentum on exiting, is now:

$$([pc/e]_\phi \; r)_6 \;=\; -\left(\frac{c\; B_{z1}\; r}{2}\right)\; r \;-\; -\left(\frac{c\; B_{z1}\; r}{2}\right)\; r \;=\; 0$$

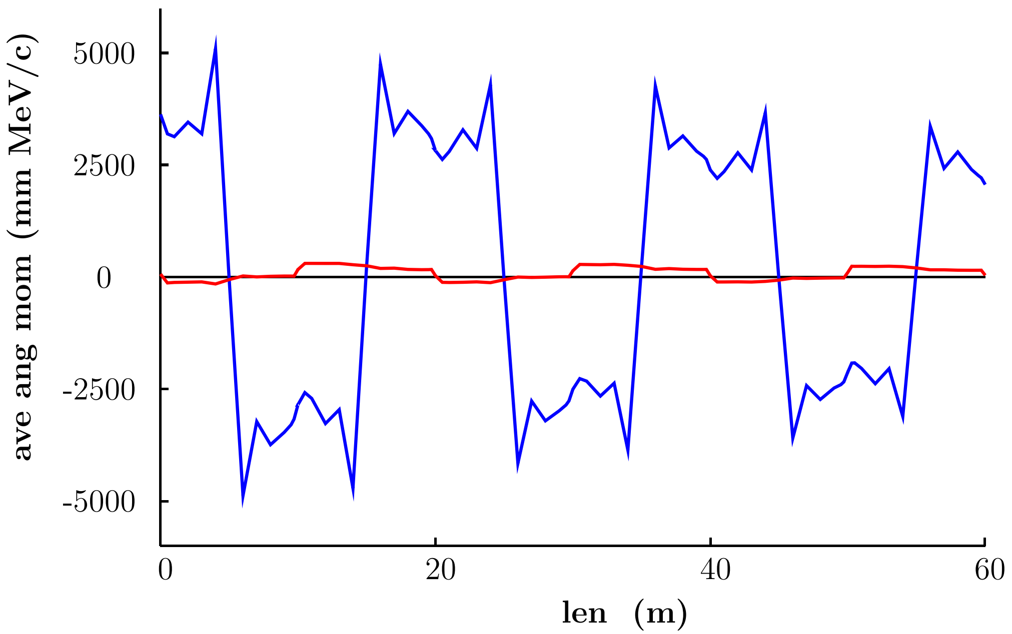
2.2.2 Example of "Single Flip" From "single flip alternative" in US Study 2



2.2.3 Alternating Solenoid Method

If we reverse the field frequently enough, no significant canonical angular momentum is developed.

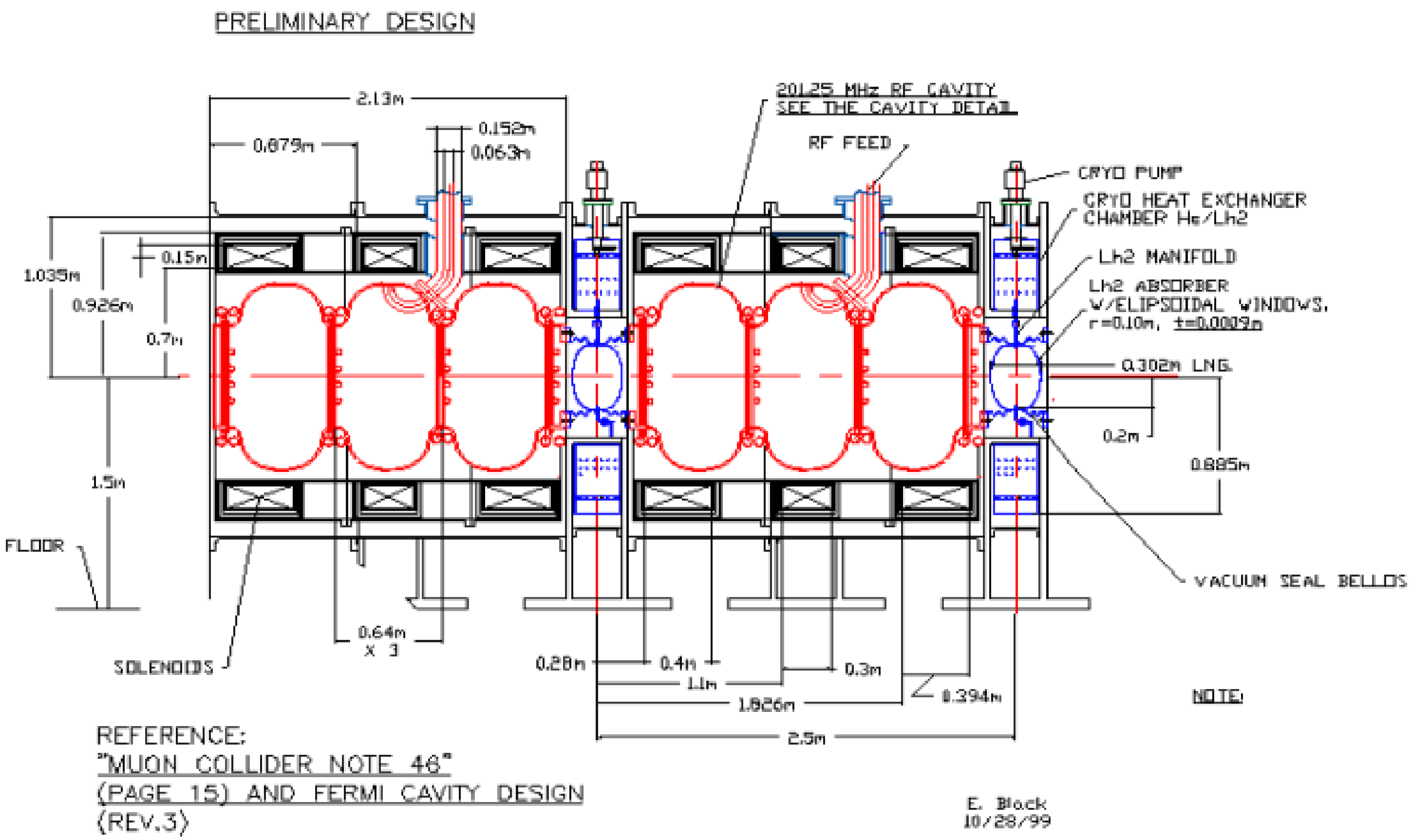
The Figure below shows the angular momenta and canonical angular momenta in a simulation of an "alternating solenoid" cooling lattice. It is seen that while the coherent angular momenta are large, the canonical angular momentum (in red) remains very small.



2.3 Focussing Lattice Designs

2.3.1 Solenoid with few "flips"

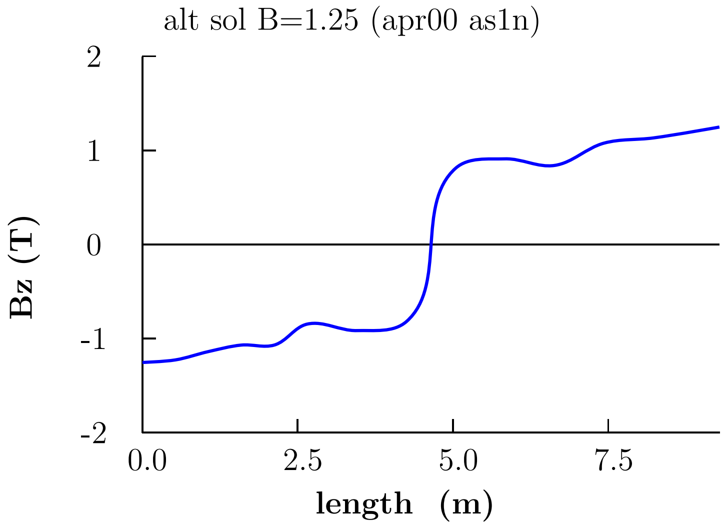
Coils Outside RF: e.g. FNAL 1 flip

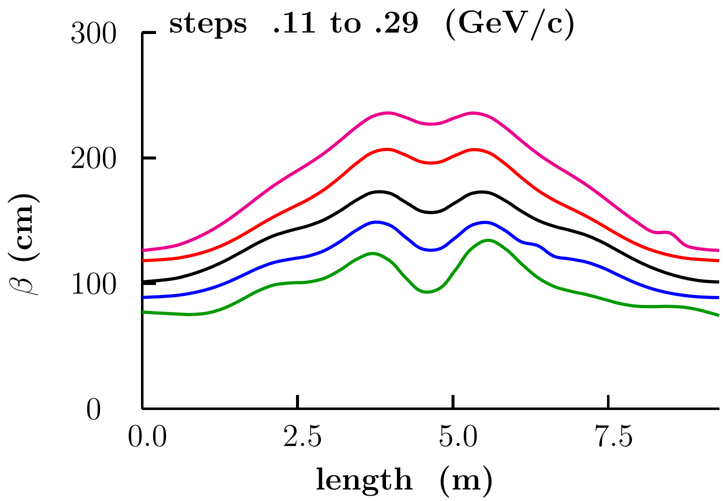


"Flips"

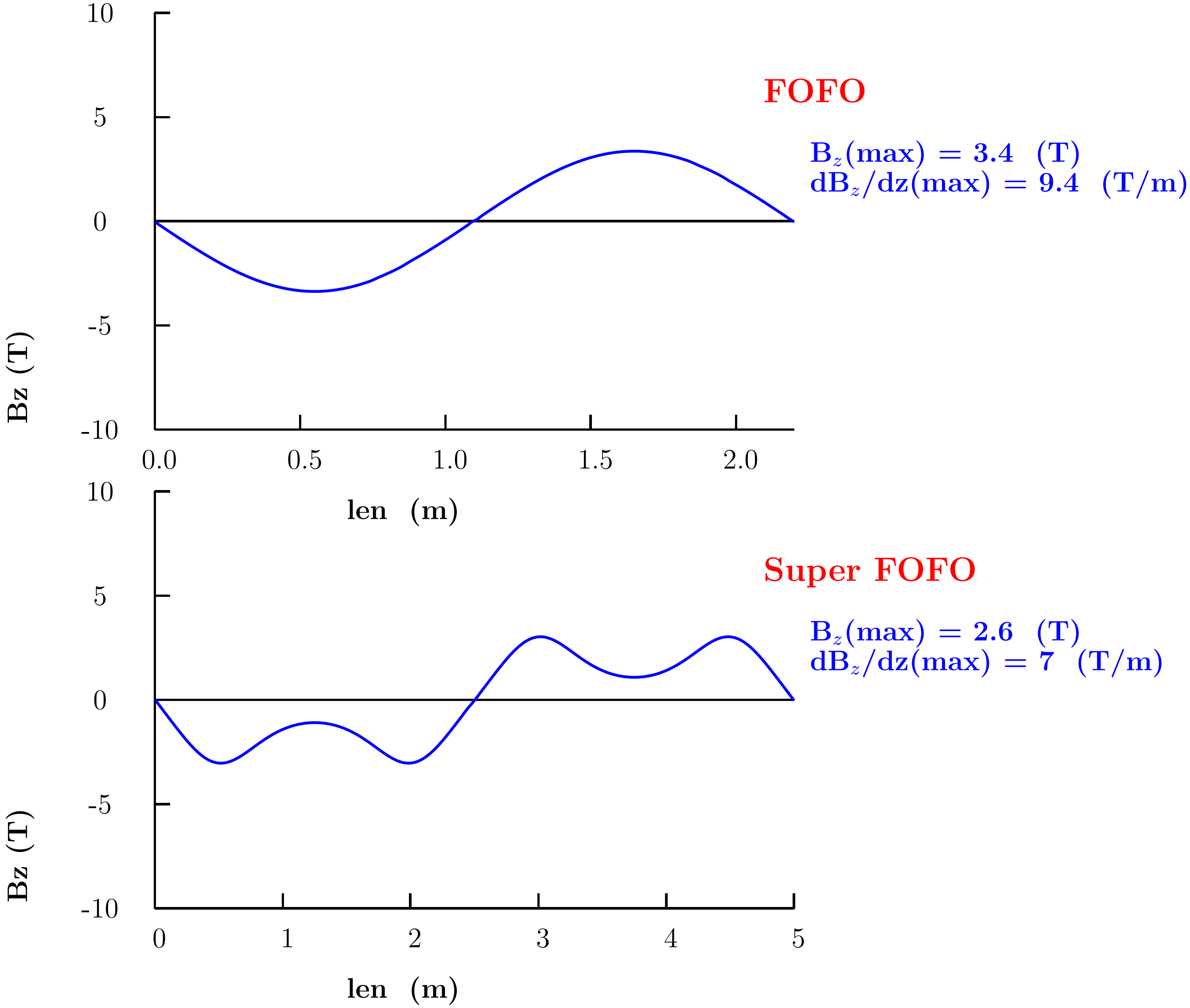
One must design the flips to match the betas from one side to the other.

For a computer designed matched flip between uniform solenoidal fields: the following figure shows B_z vs. z and the $\beta_\perp$'s vs. z for different momenta.





2.3.2 Lattices with many "flips"

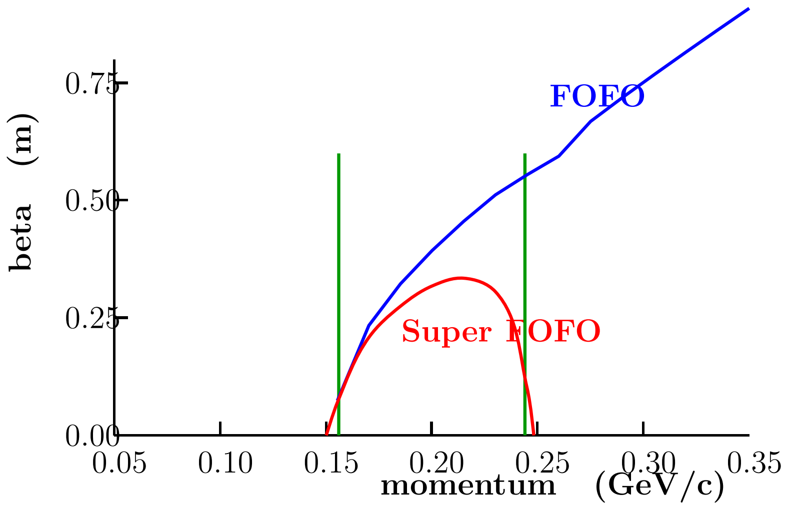


Determination of lattice betas

- Track single near paraxial particle through many cells
- plot θ_x vs x after each cell
- fit ellipse: $\beta_{x,y}$ = A((x) / A(θ_x)

beta vs. Momentum

Note "stop bands" where particles are not transmitted

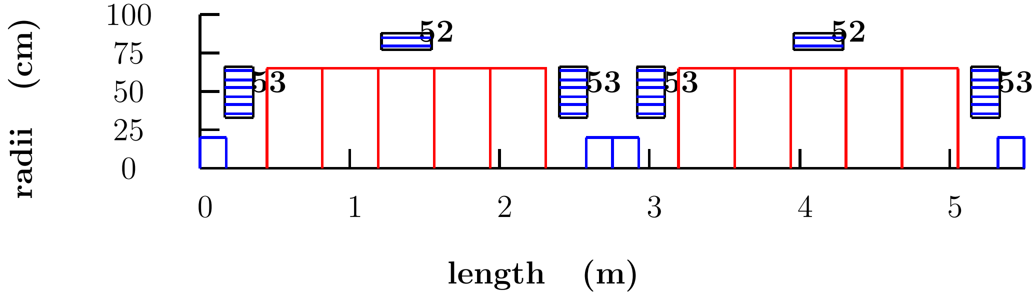


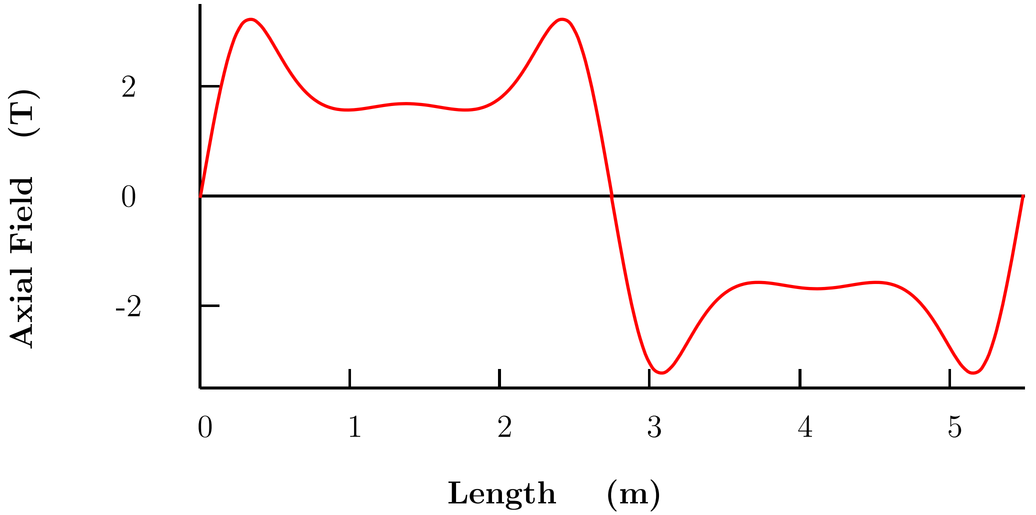
- Alternating Solenoid has largest p acceptance
- FOFO shows $\beta \propto dp/p$
- SFOFO more complicated, and better

2.3.3 Example of Multi-flip lattice

US Study 2 Super FOFO

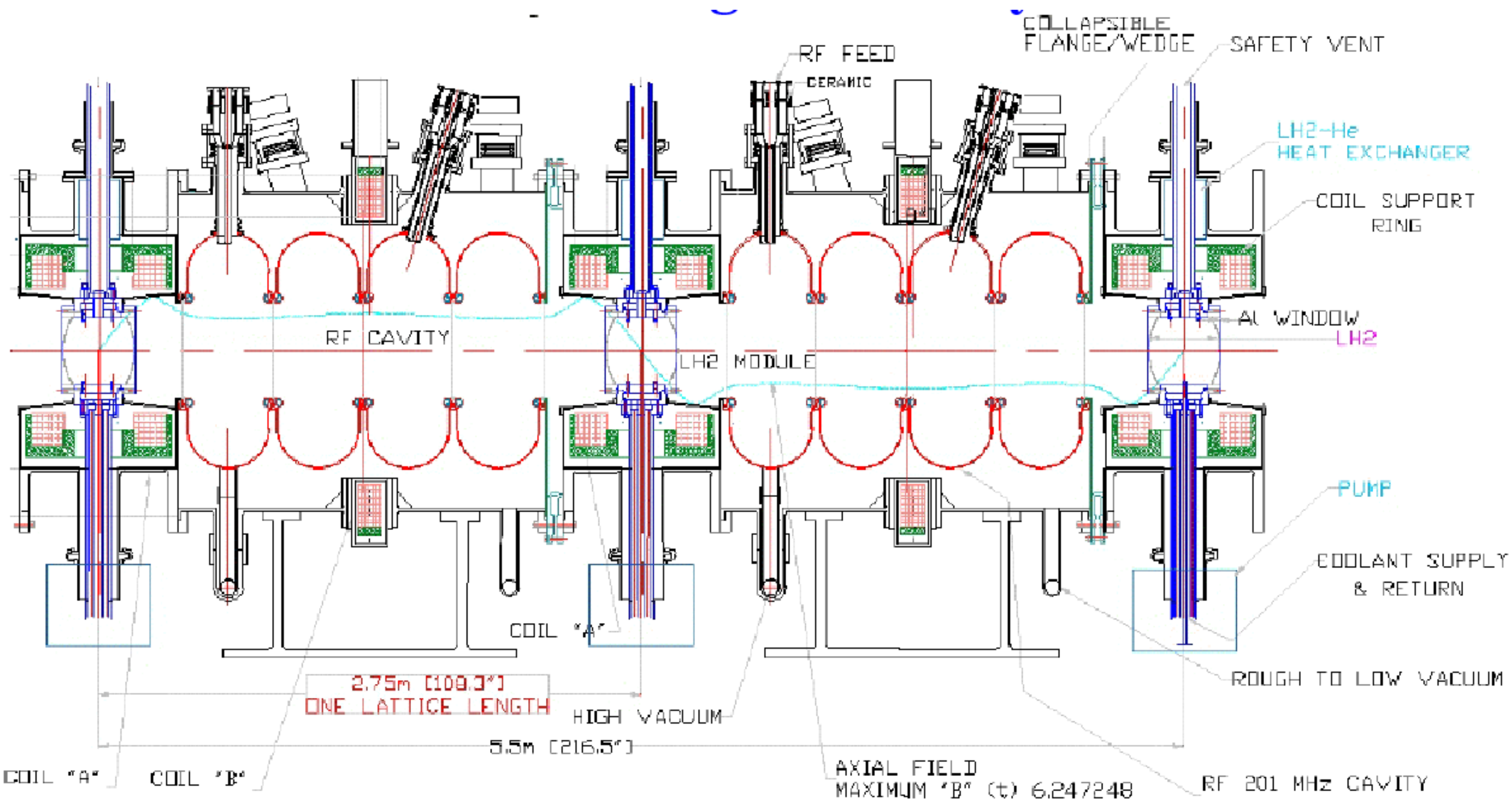
Smaller Stored E than continuous solenoid outside the RF ($\approx 1/5$)





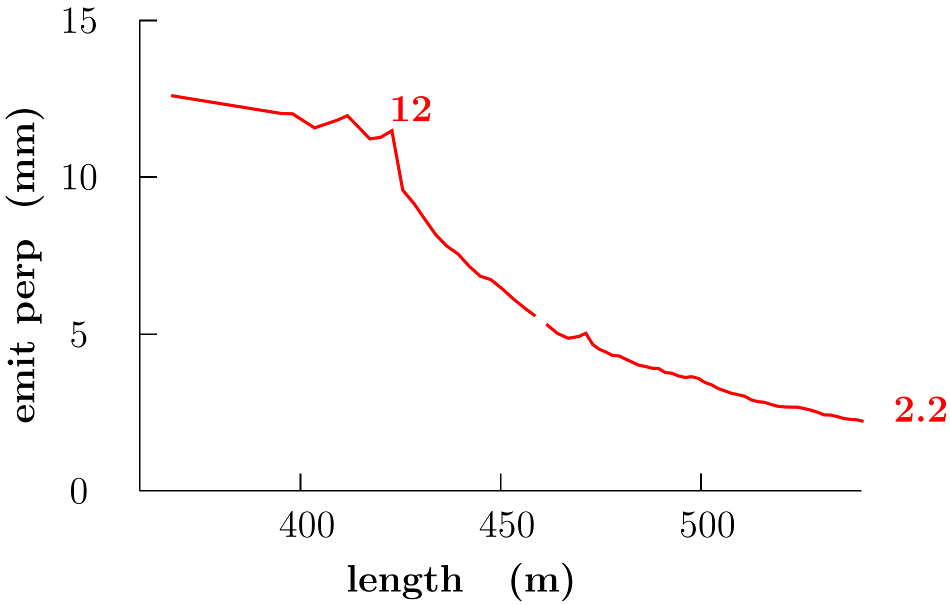
2.3.4 Hardware

At Start of Cooling



- This is the lattice to be tested in Muon Ionization Cooling Experiemnt (MICE) at RAL
- In study 2 the lattice is modified vs. length to lower $\beta_\perp$ as ϵ falls
  This keeps σ_θ and ϵ/ϵ_o more or less constant, thus maintains cooling rate

2.3.5 Study 2 Performance



With RF and Hydrogen Windows, $C_o \approx 45\ 10^{-4}$
$\beta_\perp$(end)=.18 m, β_v(end)=0.85, So

$$\epsilon_\perp(\mathrm{min}) = \frac{45\ 10^{-4}\ 0.18}{0.85} = 0.95 \ \ (\pi mm\ mrad)$$

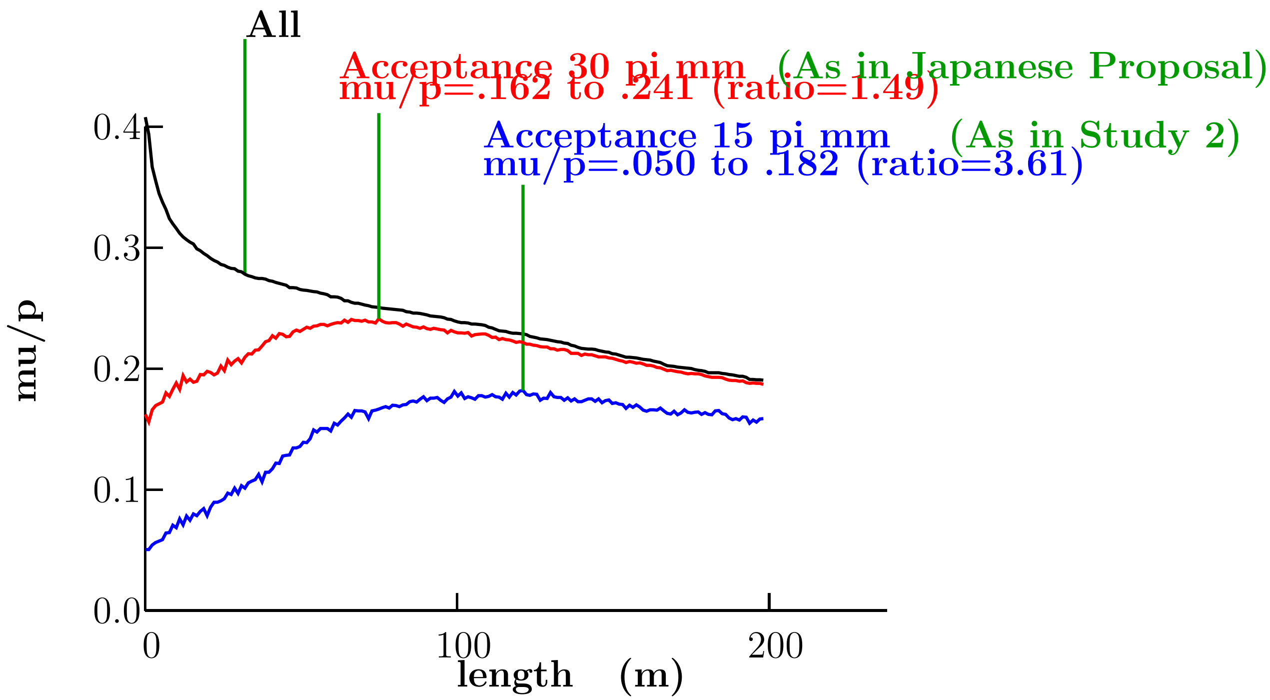
$$\frac{\epsilon_\perp}{\epsilon_\perp(\mathrm{min})} \approx 2.3$$

so from eq. 21

$$\frac{d\epsilon}{\epsilon}(\mathrm{end}) = \left(1 - \frac{\epsilon}{\epsilon(\mathrm{min})}\right)\frac{dp}{p} \approx 0.57\,\frac{dp}{p}$$

2.4 Mu/p with Cooling vs Accelerator Trans Acceptance

Using input from Study-2 Front-End (includes some mini-cooling)



- Performance at 30 pi mm without cooling
  ≈ Performance at 15 pi mm with cooling
- Not a new idea:
  Mori at KEK has proposed no cooling for a long time
- Cost of acceptance 15→30 pi mm may be less than for cooling
- If no cooling required, less R&D required for Neutrino Factory
- But we still need cooling rings for a Muon Collider

3 Longitudinal Cooling

Following the convention for synchrotron cooling we define partition functions:

$$J_{x,y,z} = \frac{\frac{\Delta\ (\epsilon_{x,y,z})}{\epsilon_{x,y,z}}}{\frac{\Delta p}{p}} \tag{27}$$

$$J_6 = J_x + J_y + J_z \tag{28}$$

where the $\Delta\epsilon$'s are those induced directly by the energy loss mechanism (ionization energy loss in this case). Δp and p refer to the loss of momentum induced by this energy loss.

In the synchrotron case, in the absence of gradients fields, $J_x = J_y = 1$, and $J_z = 2$.

In the ionization case, as we shall show, $J_x = J_y = 1$, but J_z is negative or small.

3.0.1 Transverse

From last lecture:

$$\frac{\Delta\sigma_{p\perp}}{\sigma_{p\perp}} = \frac{\Delta p}{p}$$

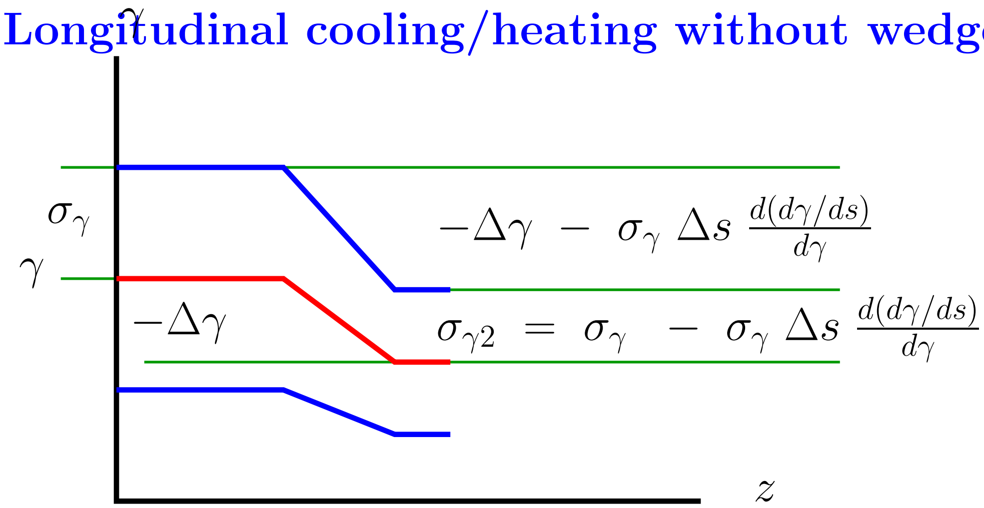
and $\sigma_{x,y}$ does not change, so

$$\frac{\Delta\epsilon_{x,y}}{\epsilon_{x,y}} = \frac{\Delta p}{p} \tag{29}$$

and thus

$$J_x = J_y = 1 \tag{30}$$

3.0.2 Longitudinal cooling/heating without wedges



The emittance in the longitudinal direction ϵ_z is (eq.5):

$$\epsilon_z \;=\; \gamma\beta_v\,\frac{\sigma_p}{p}\,\sigma_z \;=\; \frac{1}{m}\,\sigma_p\sigma_z \;=\; \frac{1}{m}\,\sigma_E\sigma_t \;=\; c\,\sigma_\gamma\,\sigma_t$$

where σ_t is the rms bunch length in time, and c is the velocity of light. Drifting between interactions will not change emittance (Louville), and an interaction will not change σ_t, so emittance change is only induced by the energy change in the interactions:

For a wedge with center thickness ℓ and height from center h ($2h\ \tan(\theta/2) = \ell$), in dispersion D ($D = \frac{dy}{dp/p}$, or with eq.2: $D = \beta_v^2\ \frac{dy}{d\gamma/\gamma}$) (see fig. above):

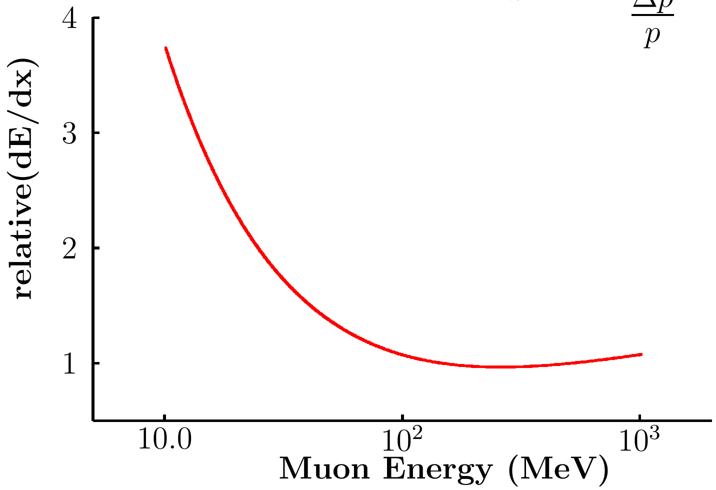
$$\frac{\Delta\epsilon_z}{\epsilon_z} \;=\; \frac{\Delta\sigma_\gamma}{\sigma_\gamma} \;=\; \frac{\sigma_\gamma\ \Delta s\ \frac{d(d\gamma/ds)}{d\gamma}}{\sigma_\gamma} \;=\; \Delta s\ \frac{d(d\gamma/ds)}{d\gamma}$$

and

$$\frac{\Delta p}{p} \;=\; \frac{\Delta\gamma}{\beta_v^2\gamma} \;=\; \frac{\ell}{\beta_v^2\gamma}\ \left(\frac{d\gamma}{ds}\right)$$

So from the definition of the partition function J_z:

$$J_z \;=\; \frac{\frac{\Delta\epsilon_z}{\epsilon_z}}{\frac{\Delta p}{p}} \;=\; \frac{\left(\Delta s\;\frac{d(d\gamma/ds)}{d\gamma}\right)}{\frac{\Delta s}{\beta_v^2\gamma}\;\left(\frac{d\gamma}{ds}\right)} \;=\; \frac{\left(\beta_v^2\;\gamma\frac{d(d\gamma/ds)}{d\gamma}\right)}{\left(\frac{d\gamma}{ds}\right)} \tag{31}$$



A typical relative energy loss as a function of energy is shown above (this example is for Lithium). It is given approximately by:

$$\frac{d\gamma}{ds} \;=\; B\;\frac{1}{\beta_v^2}\;\left(\frac{1}{2}\;ln(A\;\beta_v^4\gamma^4\;-\;\beta_v^2\right) \tag{32}$$

where

$$A \;=\; \frac{(2m_ec^2/e)^2}{I^2} \qquad B \approx \frac{0.0307}{(m_\mu c^2/e)}\;\frac{Z}{A} \tag{33}$$

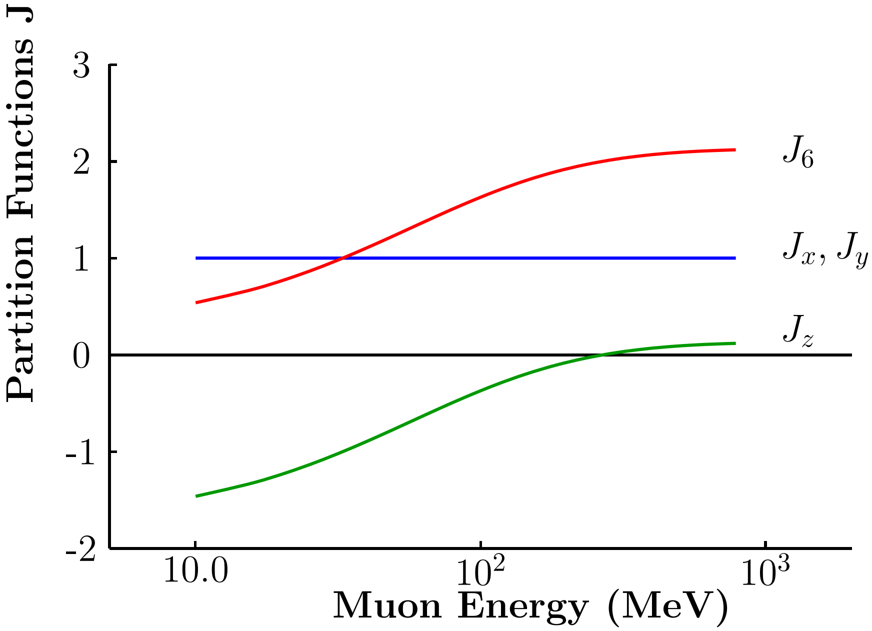
where Z and A are for the nucleus of the material, and I is the ionization potential for that material.

Differentiating the above:

$$\frac{\delta(d\gamma/ds)}{\delta\gamma} \;=\; \frac{B}{\beta_v}\;\left(\frac{2}{\beta_v\gamma}\;-\;\frac{1}{(\beta_v\gamma)^3}\;ln(A\;\beta_v^4\gamma^4)\;+\;\frac{2}{(\beta_v\gamma)^3}\right)$$

Substituting this into equation 31:

$$J_z(\text{no wedge}) \;\approx\; -\frac{\left(\frac{2}{\beta_v\gamma} \;-\; \frac{1}{(\beta_v\gamma)^3}\; ln(A\;\beta_v^4\gamma^4) \;+\; \frac{2}{(\beta_v\gamma)^3}\right)}{\left(\frac{1}{2}\; ln(A\;\beta_v^4\gamma^4 \;-\; \beta_v^2\right)}\;\beta_v^3\gamma \tag{34}$$



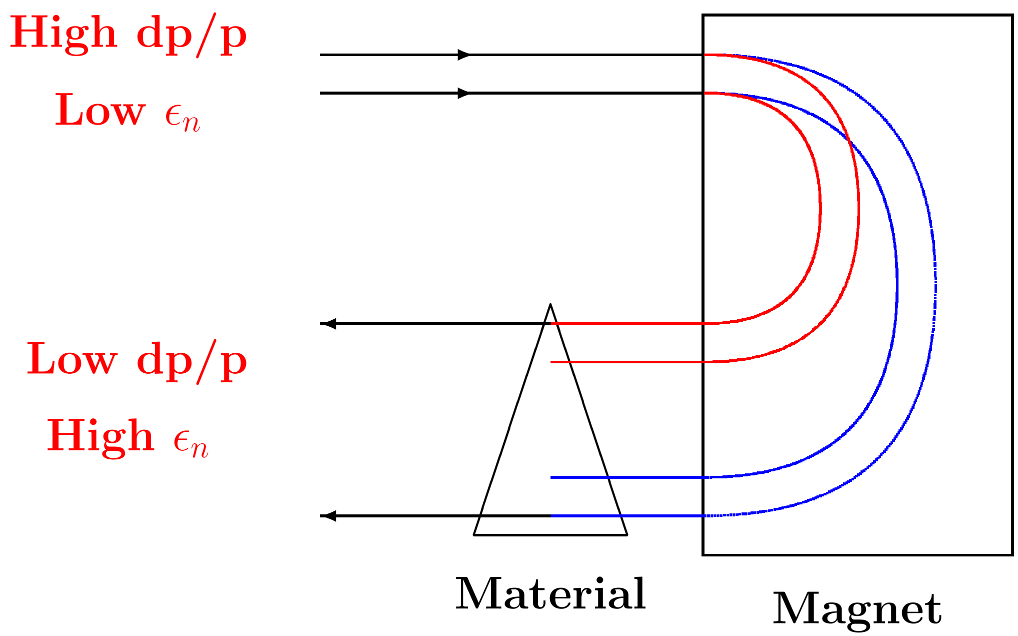
It is seen that J_z is strongly negative at low energies (longitudinal heating), ans is only barely positive at momenta above 300 MeV/c. In practice there are many reasons to cool at a moderate momentum around 250 MeV/c, where $J_z \;\approx\; 0$. However, the 6D cooling is still strong $J_6 \approx 2$.
What is needed is a method to echange cooling between the transvers and longitudinal direction s. This is done in synchroton cooling if focusing and bending is combined, but in this case, and in general, one can show that such mixing can only increase one J at the expence of the others: J_6 is conserved.

$$\Delta J_x + J_x + J_x = 0 \tag{35}$$

and for typical operating momenta:

$$J_x \;+\; J_y \;+\; J_z \;=\; J_6 \;\approx\; 2.0 \tag{36}$$

3.0.3 Emittance Exchange

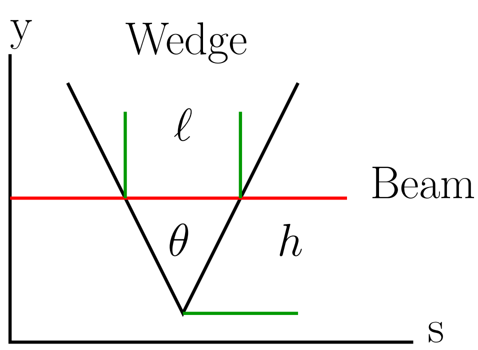


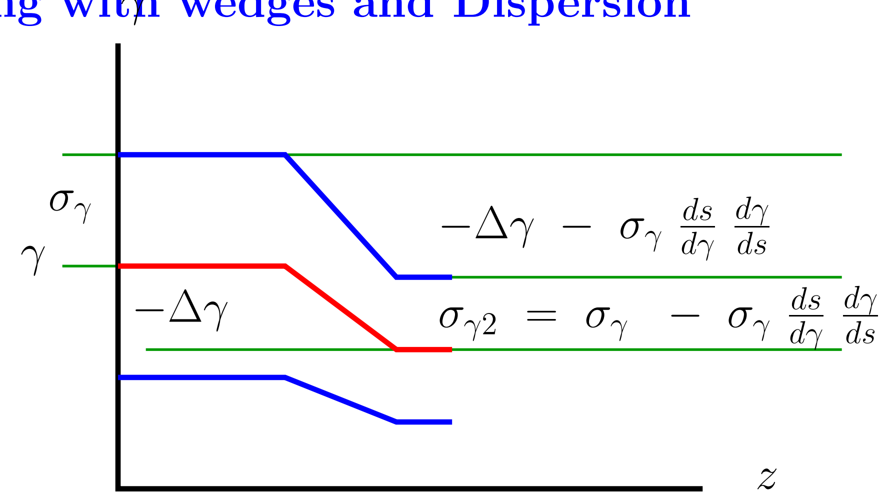
- dp/p reduced
- But σ_y increased

- Long Emittance reduced
- Trans Emittance Increased

- "Emittance Exchange"

3.0.4 Longitudinal cooling with wedges and Dispersion





For a wedge with center thickness ℓ and height from center h ($2h \; \tan(\theta/2) = \ell$), in dispersion D ($D = \frac{dy}{dp/p}$, or with eq.2: $D = \beta_v^2 \, \frac{dy}{d\gamma/\gamma}$) (see fig. above):

$$\frac{\Delta\epsilon_z}{\epsilon_z} = \frac{\Delta\sigma_\gamma}{\sigma_\gamma} = \frac{\sigma_\gamma \, \frac{ds}{d\gamma} \left(\frac{d\gamma}{ds}\right)}{\sigma_\gamma} = \frac{ds}{d\gamma}\left(\frac{d\gamma}{ds}\right) = \left(\frac{\ell}{h}\right) \frac{D}{\beta_v^2 \, \gamma} \left(\frac{d\gamma}{ds}\right)$$

and

$$\frac{\Delta p}{p} = \frac{\Delta\gamma}{\beta_v^2 \gamma} = \frac{\ell}{\beta_v^2 \gamma} \left(\frac{d\gamma}{ds}\right)$$

So from the definition of the partition function J_z:

$$J_z(\text{wedge}) = \frac{\frac{\Delta\epsilon_z}{\epsilon_z}}{\frac{\Delta p}{p}} = \frac{\left(\frac{\ell}{h}\right) \frac{D}{\beta_v^2 \, \gamma} \left(\frac{d\gamma}{ds}\right)}{\frac{\ell}{\beta_v^2 \gamma} \left(\frac{d\gamma}{ds}\right)} = \frac{D}{h} \qquad (37)$$

$$J_z = J_z(\text{no wedge}) + J_z(\text{wedge}) \qquad (38)$$

But from eq.35, for any finite J_z(wedge), J_x or J_y will change in the opposite direction.

3.0.5 Longitudinal Heating Terms

Since $\epsilon_z = \sigma_\gamma \, \sigma_t \, c$, and t and thus σ_t is conserved in an interaction

$$\frac{\Delta \epsilon_z}{\epsilon_z} = \frac{\Delta \sigma_\gamma}{\sigma_\gamma}$$

Straggling, from Perkins text book, converted to MKS:

$$\Delta(\sigma_\gamma) = \frac{\Delta \sigma_\gamma^2}{2\sigma_\gamma} \approx \frac{1}{2\sigma_\gamma} \, 0.06 \, \frac{Z}{A} \left(\frac{m_e}{m_\mu}\right)^2 \gamma^2 \left(1 - \frac{\beta_v^2}{2}\right) \rho \, \Delta s$$

From eq. 2: $\Delta E = E\beta_v^2 \, \frac{\Delta p}{p}$, so:

$$\Delta s = \frac{\Delta E}{dE/ds} = \frac{1}{dE/ds} E \, \beta_v^2 \, \frac{\Delta p}{p}$$

so

$$\frac{\Delta \epsilon_z}{\epsilon_z} = \frac{0.06}{2\sigma_\gamma^2} \, \frac{Z}{A} \left(\frac{m_e}{m_\mu}\right)^2 \gamma^2 \left(1 - \frac{\beta_v^2}{2}\right) \rho \, \frac{\beta_v^2 \, E}{dE/ds} \, \frac{\Delta p}{p}$$

This can be compared with the cooling term

$$\frac{\Delta \epsilon_z}{\epsilon_z} = - \, J_z \, \frac{dp}{p}$$

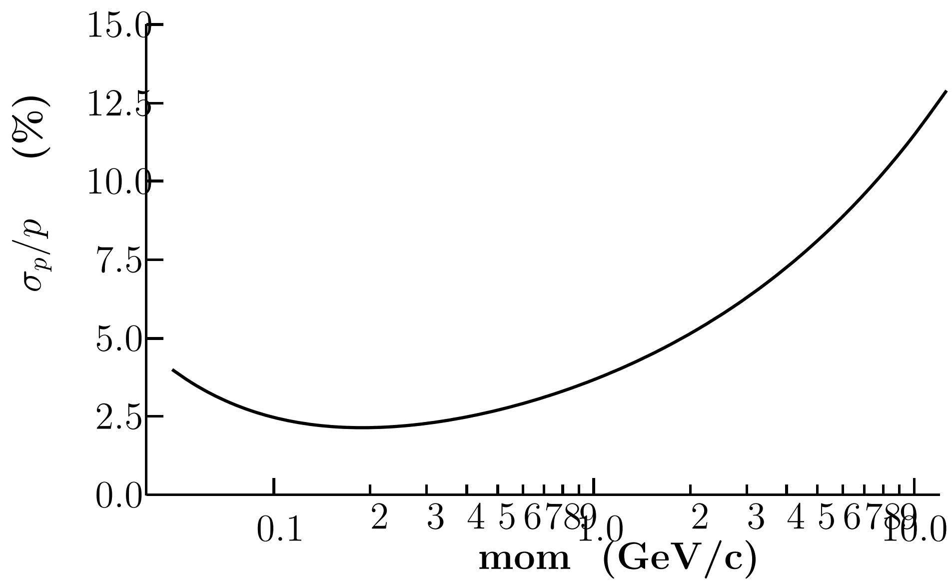
giving an equilibrium:

$$\frac{\sigma_p}{p} = \left(\left(\frac{m_e}{m_\mu}\right) \sqrt{\frac{0.06 \; Z \; \rho}{2 \; A \; (d\gamma/ds)}}\right) \sqrt{\frac{\gamma}{\beta_v^2} \left(1 - \frac{\beta_v^2}{2}\right) \quad \frac{1}{J_z}} \tag{39}$$

For Hydrogen, the value of the first parenthesis is ≈1.36 %.

Without coupling, J_z is small or negative, and the equilibrium does not exist. But with equal partition functions giving $J_z \approx 2/3$ then this expression, for hydrogen, gives: the values ploted below.

The following plot shows the dependency for hydrogen



It is seen to favor cooling at around 200 MeV/c, but has a broad minimum.

3.1 Emittance Exchange Studies

- Attempts at separate cooling & exch.
  - Wedges in Bent Solenoids
  - Wedges in Helical Channels[1]

  Poor performance & problems matching between them

- Attempts in rings with alternate cooling & exchange
  - Balbakov[2] with solenoid focus
    achieved Merit=90

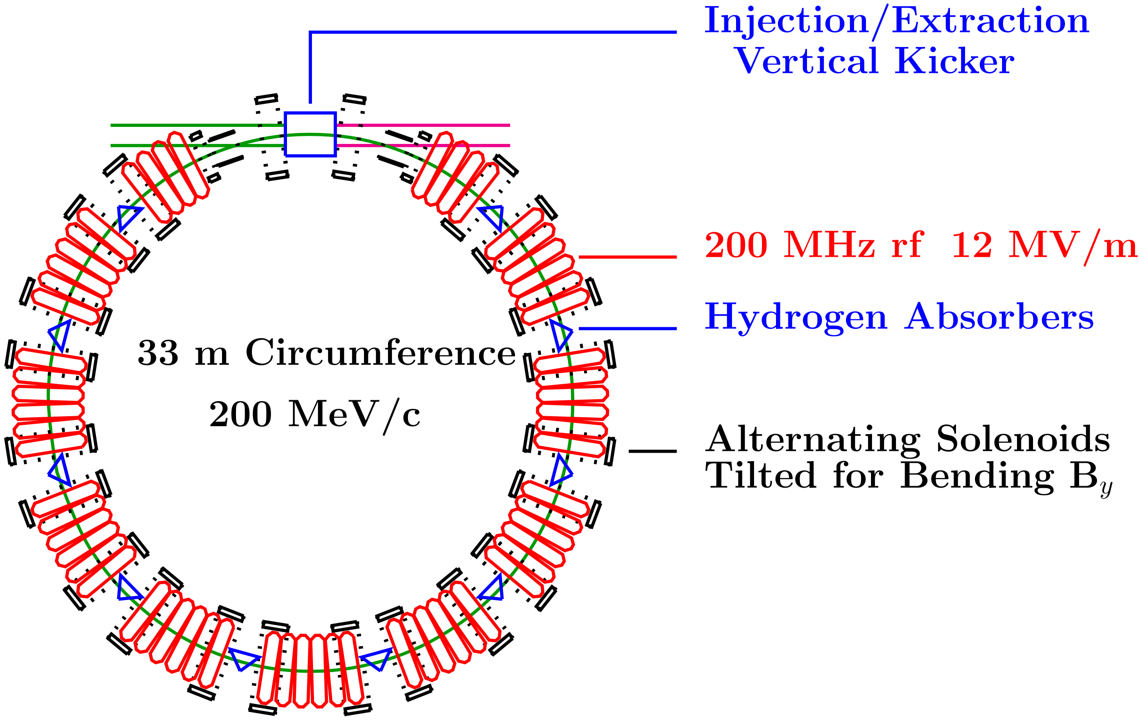
- Attempts in rings with combined cooling & exchange
  - Garren et al[3] Quadrupole focused ring
    achieved Merit ≈15
  - Garren et al: Bend only focusing
    achieved Merit ≈100
  - Palmer et al[4]
    achieved Merit ≈140

[1] MUC-146, 147, 187, & 193
[2] MUC-232 & 246
[3] Snowmass Proc.
[4] MUC-239

3.2 Example RFOFO Ring

3.2.1 Introduction

R.B. Palmer R. Fernow J. Gallardo[5], and Balbekov[6]

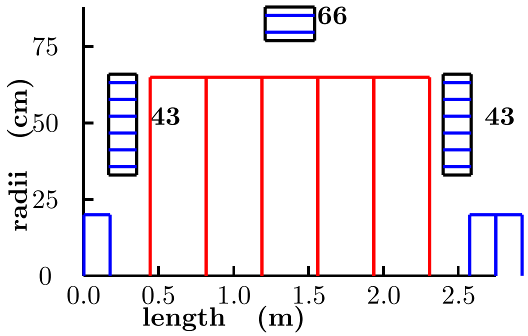


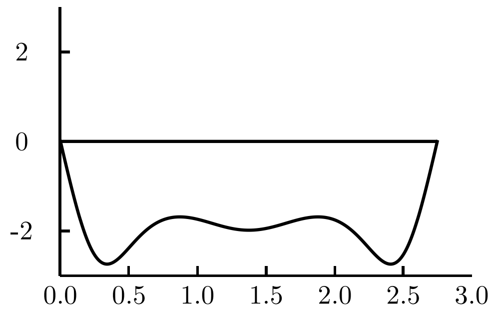
[5]Fernow and others: MUC-232, 265, 268, & 273

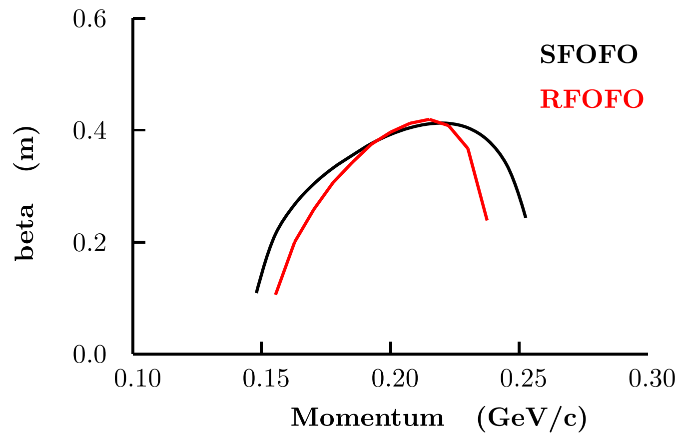
[6]V.Balbekov "Simulation of RFOFO Ring Cooler with Tilted Solenoids" MUC-CONF–0264

3.2.2 Lattice

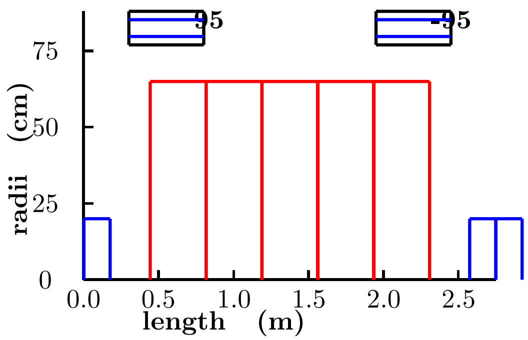
SFOFO as in Study 2

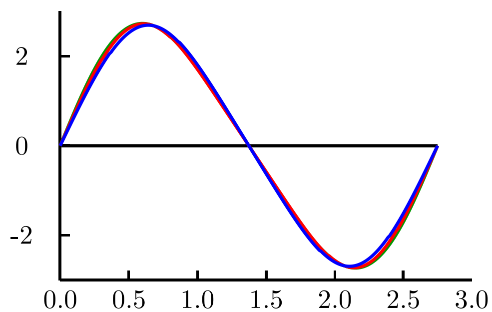






RFOFO has Reversed Fields



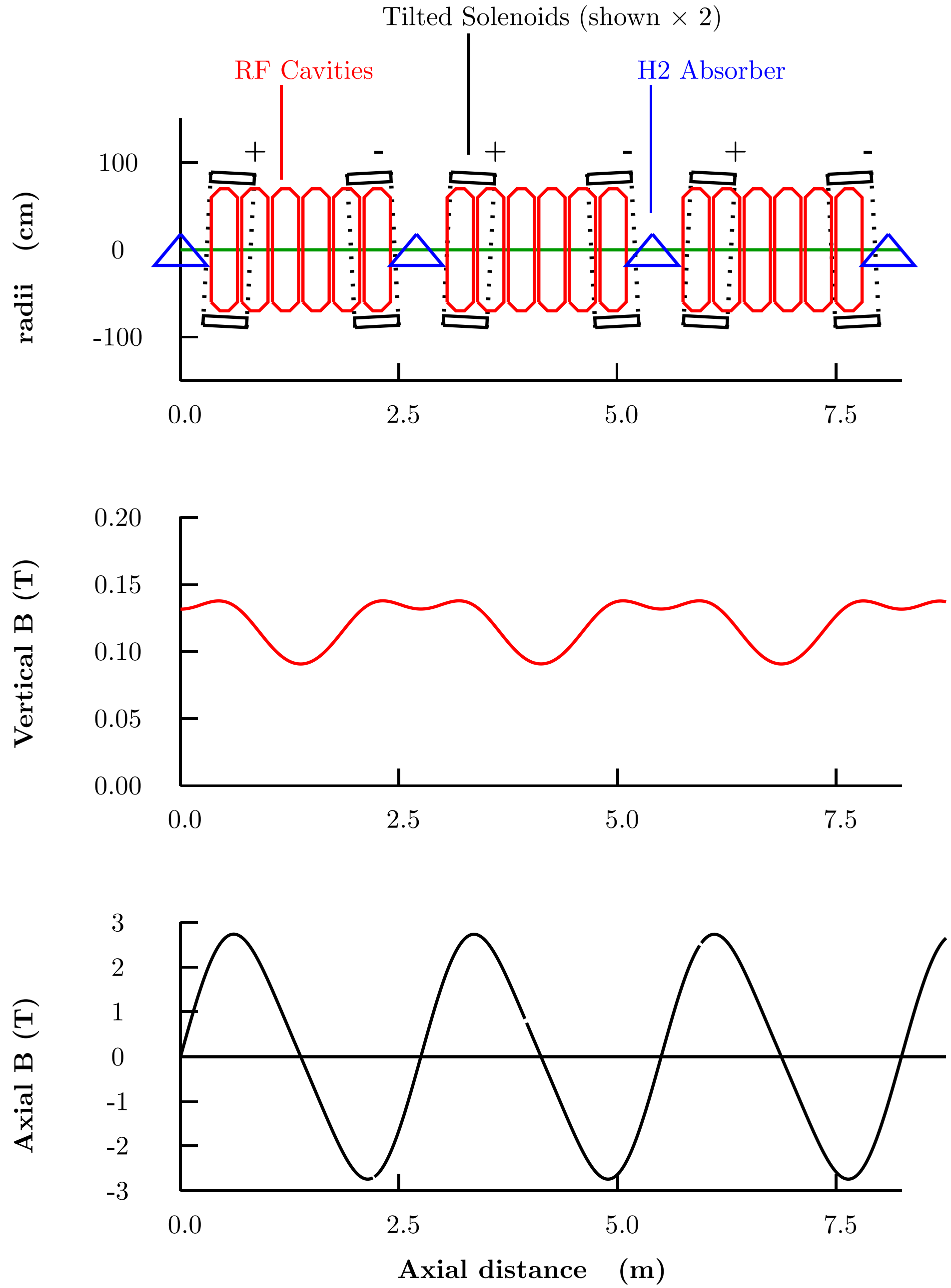


RFOFO chosen

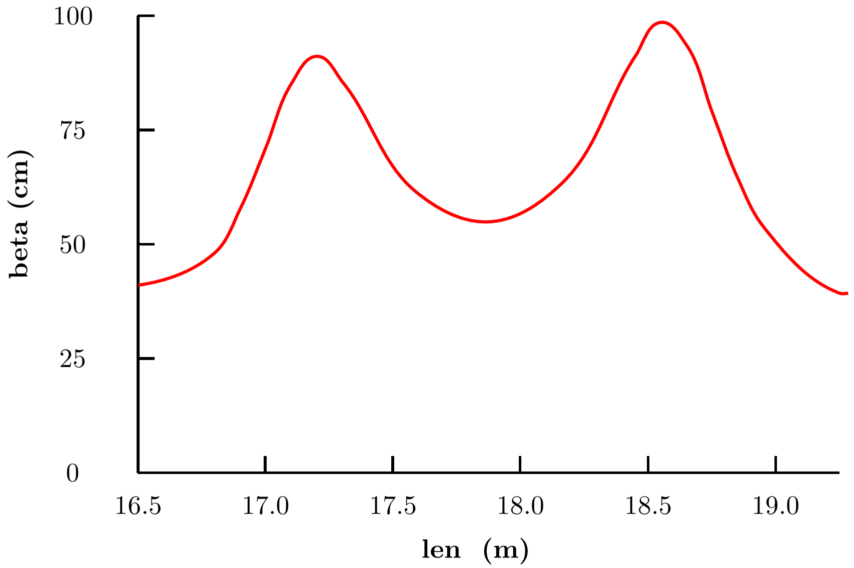
- Less Mom acceptance
  BUT
- All cells the same $\longrightarrow$ Fewer resonances

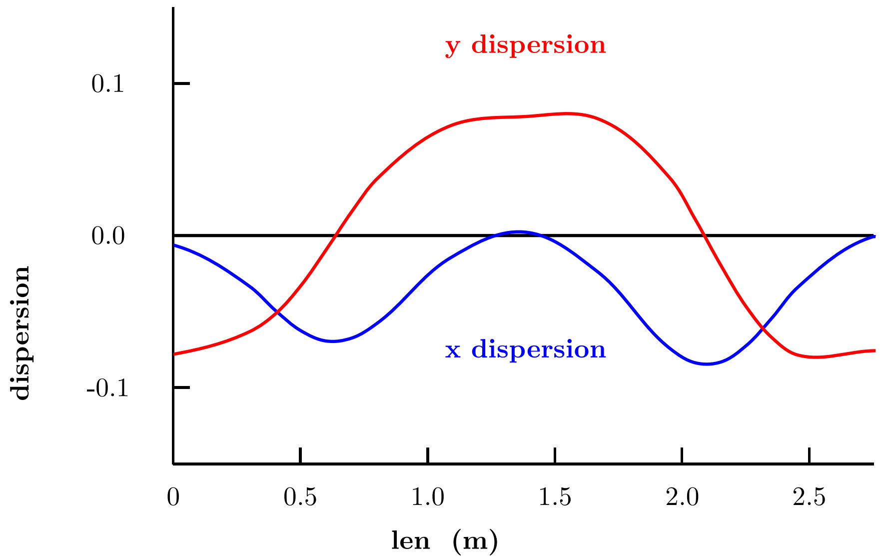
3.2.3 Coil Layout

Tilt Coils to get Bend



3.2.4 Beta and Dispersion





Dispersion is rotating back and forth

3.2.5 Params for Simulation

Coils

gap	start	dl	rad	dr	tilt	I/A
m	m	m	m	m	rad	A/mm^2
0.310	0.310	0.080	0.300	0.200	0.0497	86.25
0.420	0.810	0.080	0.300	0.200	0.0497	86.25
0.970	1.860	0.080	0.300	0.200	-.0497	-86.25
0.420	2.360	0.080	0.300	0.200	-.0497	-86.25

amp turns 5.52 (MA)
amp turns length 13.87326 (MA m)
cell length 2.750001 (m)

Wedge

Material		H2
Windows		none
Radius	cm	18
central thickness	cm	28.6
min thickness	cm	0
wedge angle	deg	100
wedge azimuth from vertical	deg	30

RF

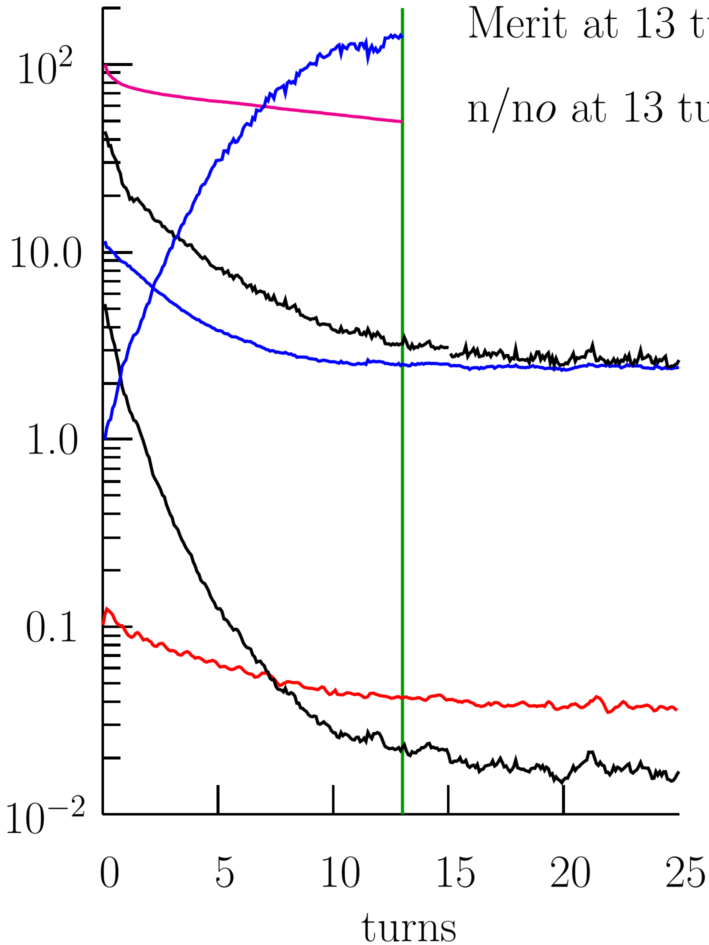
Cavities		6
Lengths	cm	28
Central gaps	cm	5
Radial aperture	cm	25
Frequency	MHz	201.25
Gradient	MV/m	16
Phase rel to fixed ref	deg	25
Windows		none

3.2.6 Performance

Using Real Fields, but no windows or injection insertion

$$\text{Merit} \;=\; \frac{n}{n_o}\,\frac{\epsilon_{6,o}}{\epsilon_6} \;=\; \frac{\text{Initial phase density}}{\text{final phase density}}$$

$n/n_o = \;1543 \;\;/\;\; 4494$



Merit at 13 turns 139 Falls after 13 turns from decay loss

n/no at 13 turns 0.50

$\epsilon \parallel$ 43.9 to 2.65 ($\pi\ mm$)
$\epsilon \perp$ 11.4 to 2.43 ($\pi\ mm$)

dp/p 10.2 to 3.6 %
$\epsilon 6$ 5.3 to 0.017 $(\pi\ mm)^3$

3.2.7 Compare with Linear theory

$D = 7$ cm, $\ell = 28.6$ cm, and

$$h = \frac{\ell}{2 \tan(100^o/2)} = 12 \quad \text{cm}$$

so

$$J_z = \frac{D}{h} = 0.58$$

Since there is good mixing between x and y so $J_x = J_y$,
and from equ 36, $\Sigma J_i \approx 2.0$, so

$$J_x = J_y \approx \frac{2 - 0.58}{2} = 0.71$$

i.e. The wedge angle was chosen to give nearly equal partition functions in all 3 coordinates, and gives the maximum merit factor.

The theoretical equilibrium emittances are now (eq.20):

$$\epsilon_\perp(\text{min}) = \frac{C\ \beta_\perp}{J\ \beta_v} = \frac{38\ 10^{-4}\ 0.4}{0.71\ 0.85} = 2.5 \ \ (\pi\ mm)$$

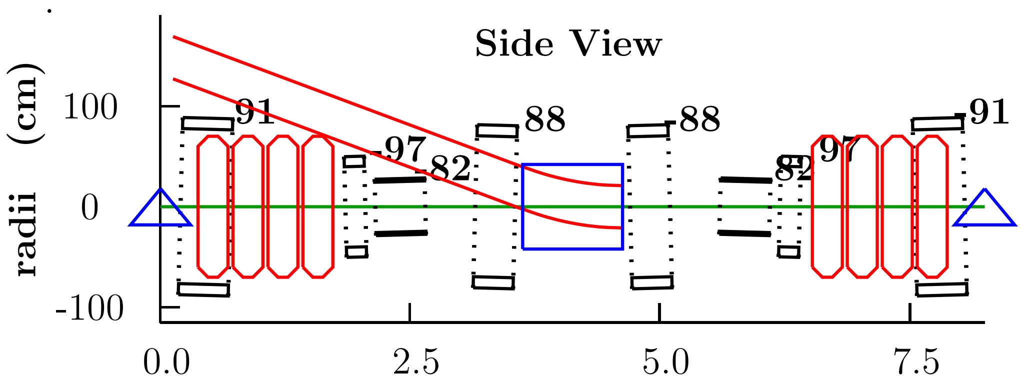
c.f. 2.43 (π mm) observed, which is very good agreement considering the approximations used.

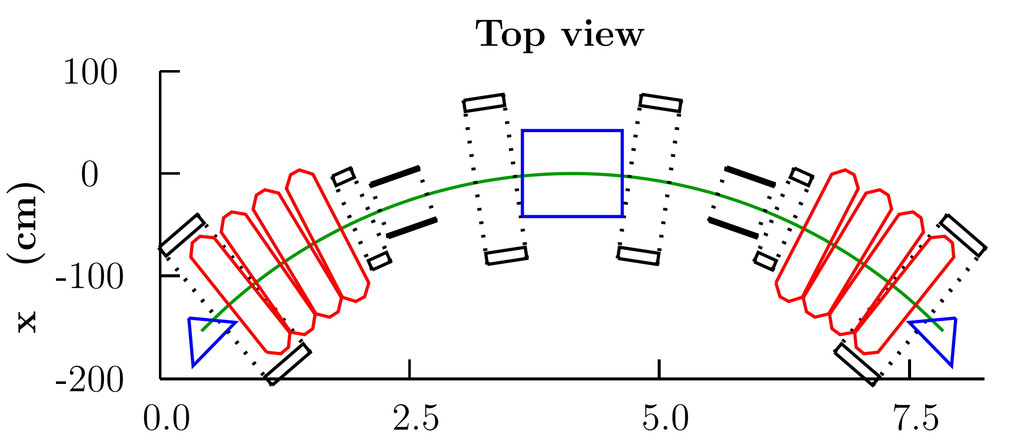
And from equation 39 we expect

$$\frac{dp}{p}(\text{min}) \ \approx 2.3\%$$

compared with 3.6% observed, which is less good agreement. This may arise from the poorer approximation of the real Landau scattering distribution by a simple gaussian.

3.2.8 Insertion for Injection/Extraction

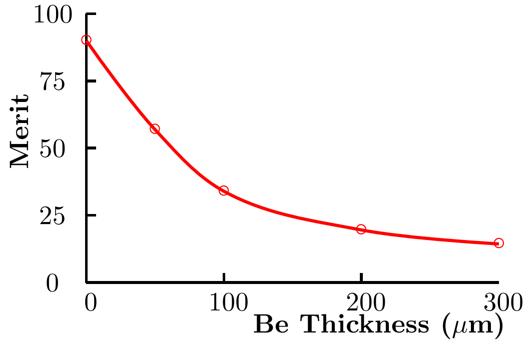




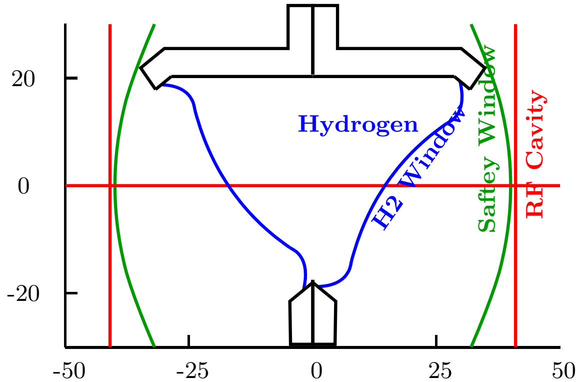
- First Simulation gave Merit = 10
  Synchrotron tune = 2.0: Integer
- Increase energy, wedge angle, and add matching.
- Merit achieved ≈ 100

3.2.9 Further Problems under study

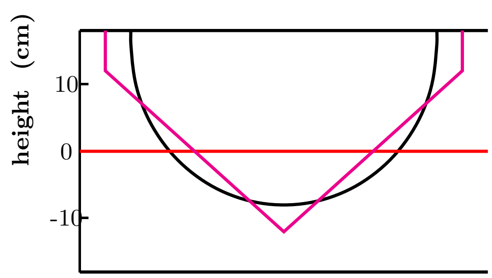
- RF windows must be very thin ($\leq$ 50 microns)
  RF at 70 deg will help



- Design of wedge absorber



For 80 Degrees

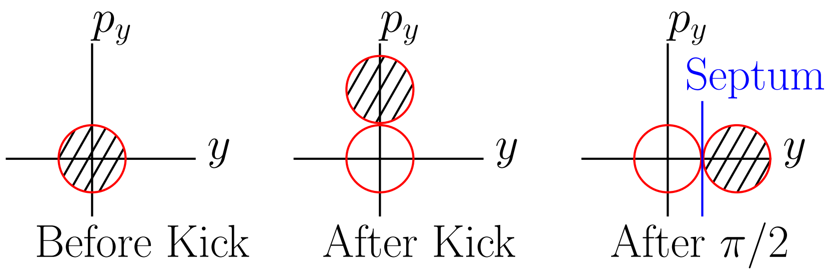


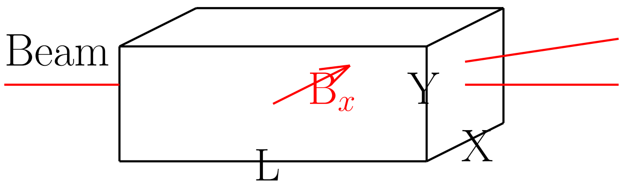
For 100 Degrees

- Absorber heating is high for many passes
- The kicker (problem common to all rings)

3.3 Kickers

3.3.1 Minimum Required kick





$$f_\sigma = \frac{\mathrm{Ap}}{\sigma} \qquad \mu = \mathrm{inf} \qquad F = \frac{Y}{X}$$

$$I = F \left(\frac{4\ f_\sigma^2\ m_\mu}{\mu_o\ c} \right) \quad \frac{\epsilon_n}{L}$$

$$V = \left(\frac{4\ f_\sigma^2\ m_\mu\ R}{c} \right) \quad \frac{\epsilon_n}{\tau}$$

$$U = F \left(\frac{m_\mu^2\ 8\ f_\sigma^4\ R}{\mu_o\ c^2} \right) \quad \frac{\epsilon_n^2}{L}$$

- muon ϵ_n $\gg$ other ϵ_n's
- So muon kicker Joules $\gg$ other kickers
- Nearest are $\bar{p}$ kickers

Compare with others

For $\epsilon_{\perp} = 10\ \pi$mm,**(Acceptance=90 pi mm)** $\beta_{\perp} = 1$m, & τ=50 nsec:
After correction for finite μ and leakage flux:

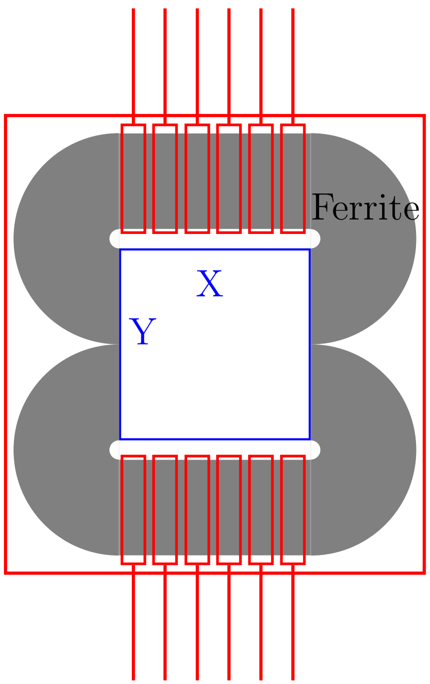
		μ Cooling	CERN $\bar{p}$	Ind Linac
$\int B d\ell$	Tm	.30	.088	
L	m	1.0	$\approx$5	5.0
$t_{\rm rise}$	ns	**50**	**90**	**40**
B	T	**.30**	**$\approx$0.018**	**0.6**
X	m	.42	.08	
Y	m	.63	.25	
$V_{\rm 1turn}$	kV	**3,970**	**800**	**5,000**
$U_{\rm magnetic}$	J	**10,450**	**$\approx$13**	**8000**

Note

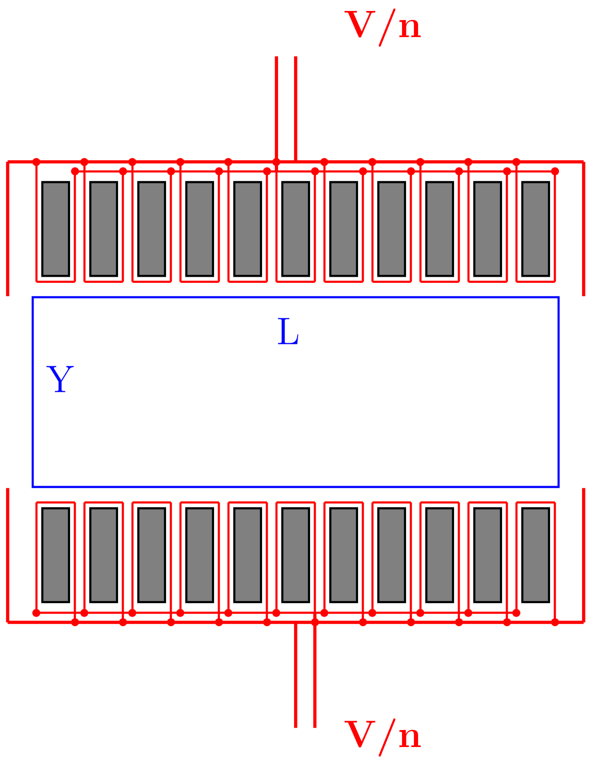
- U is 3 orders above $\bar{p}$, and 1 order of magnitude more than 30 pi mm FFAG
- Same order as Induction
- And t same order as a few m of induction linac
- But V is too High for single turn kicker

3.3.2 Induction Kicker

- Drive Flux Return
- Subdivide Flux Return Loops
  Solves Voltage Problem
- Conducting Box Removes
  Stray Field Return



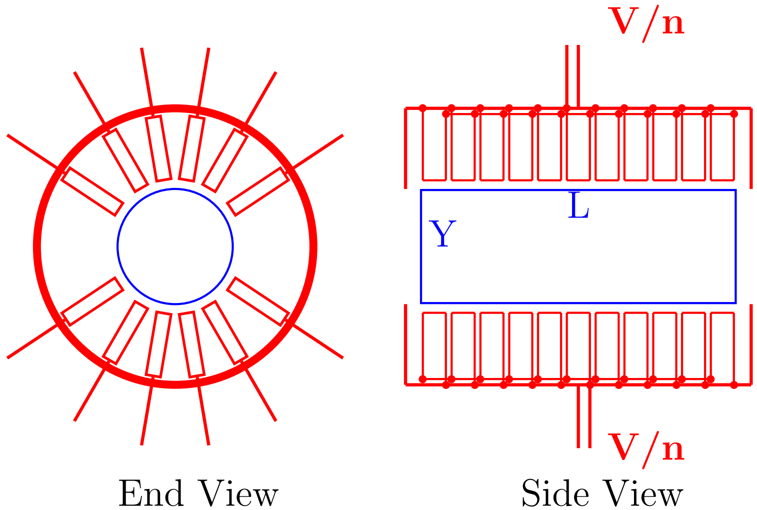
End View



Side View

Works with no Ferrite

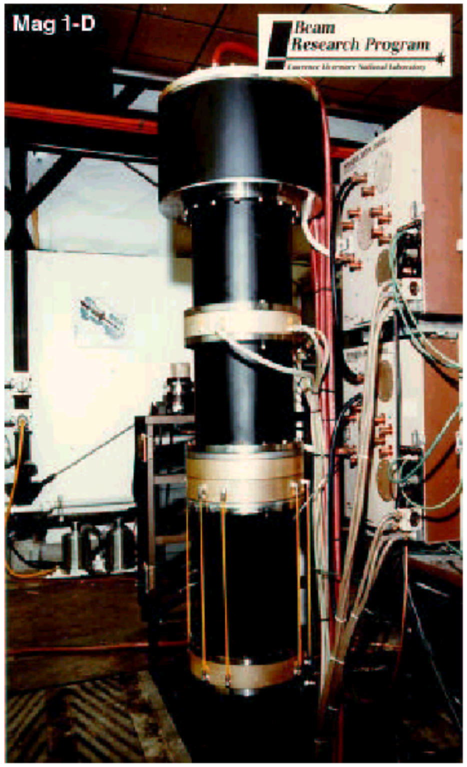
- V = the same
- U$\approx$ 2.25$\times$
- I$\approx$ 2.25$\times$
- No rise time limit
- Not effected by solenoid fields

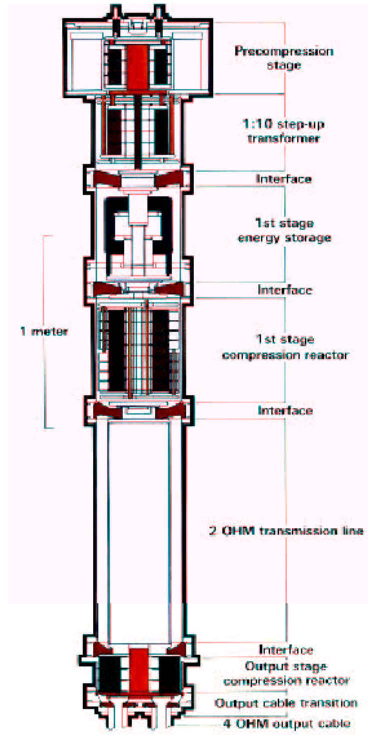


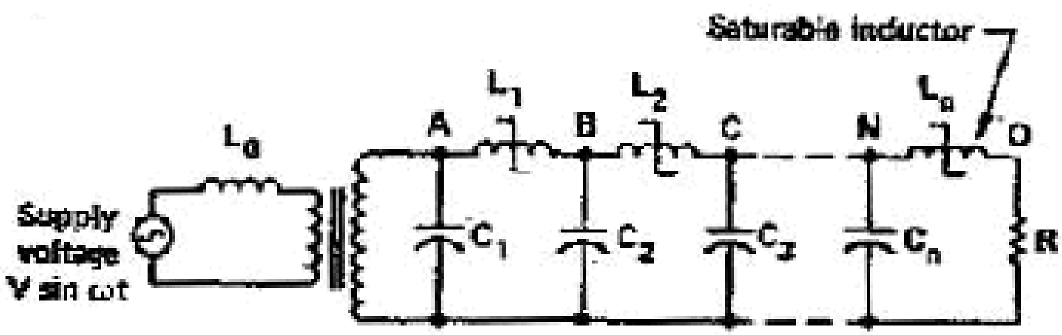
- If non Resonant: 2 Drivers
  for inj. & extract.
  Need 24 $\times$2 Magamps ($\approx$ 20 M$)
- If Resonant: 1 Driver, 2$\times$efficient
  Need 12 Magamps ($\approx$ 5 M$)

3.3.3 Magnetic Amplifiers

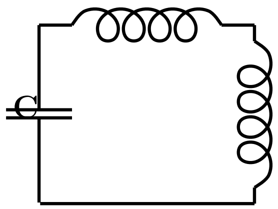
Used to drive Induction Linacs
similar to ATA or DARHT







Magamp principle



Initially Unsaturated, $L = L_1$ is large:

$$\tau_L \;=\; \sqrt{(L+L_1)C} \quad \text{is slow}$$

The current I rises slowly:

$$I \;=\; I_o \, \sin\left(\frac{t}{\tau_L}\right)$$

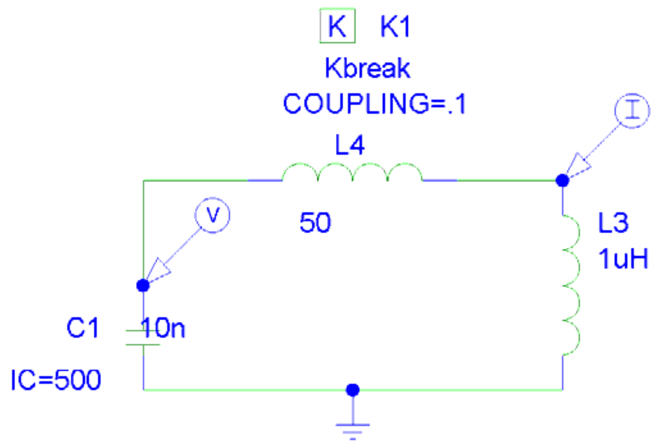
When the inductor saturates
$L = L_2$ is small:

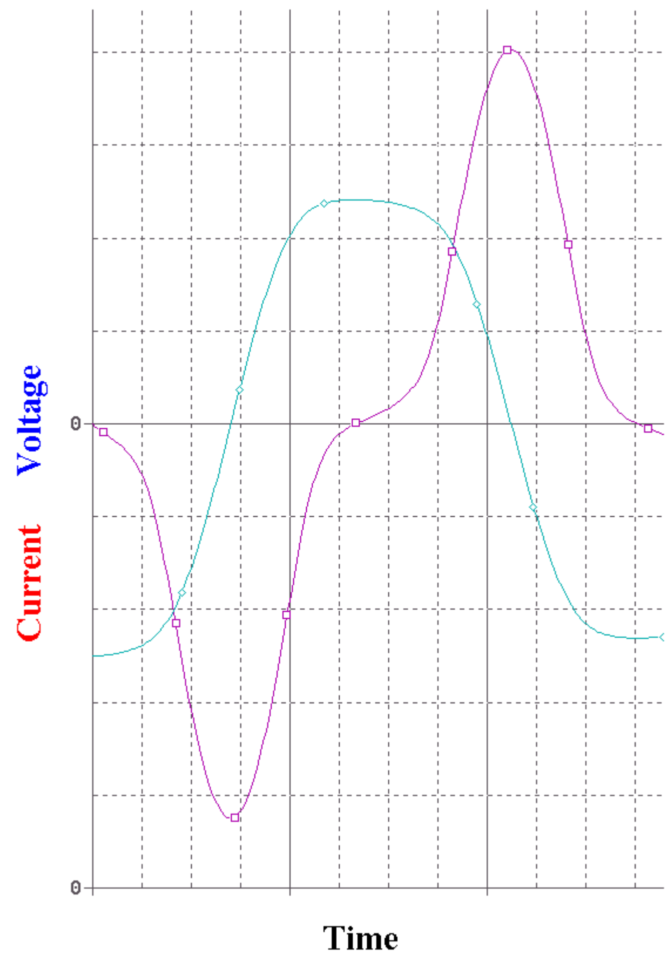
$$\tau_S \;=\; \sqrt{(L+L_2)C} \qquad \text{is fast}$$

After approx π phase
Inductor regains its high inductance
The oscillation slows before reversing.

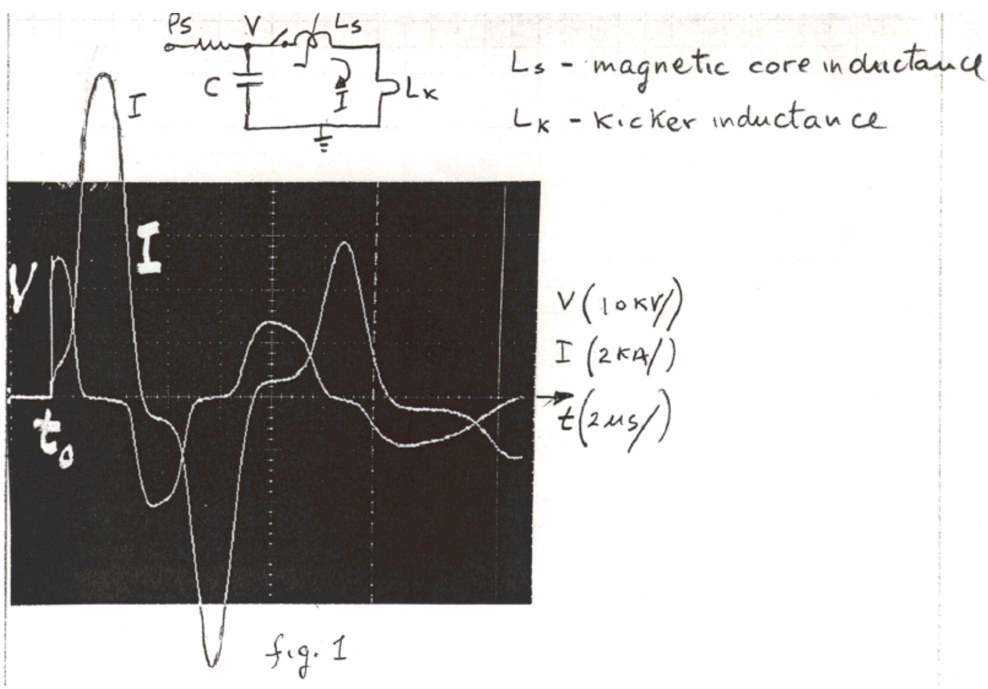
Pspice Simulation

a) Single stage





Circuit Model (Reginato)



3.4 Ring Cooler Conclusion

- Need for very thin windows is greater than for linear coolers
- Work needed on Hydrogen wedge design
- Much Work needed on Insertion
  but probably doable
- The Kicker is the least certain
- Needs shorter bunch train than linear
  $\longrightarrow$ greater dp/p or less acceptance
  $\longrightarrow$ Worse performance
- Needed for Muon Collider, but maybe not useful for Neutrino Factory