Harmonic Number Jump in a Ring Filled with Cavities Distributed Everywhere

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10 November 2006

Ring Filled with Cavities and Harmonic Number Jump

- Harmonic number jump works well with a single cavity in a ring
- Muon FFAGs need the ring filled with cavities
- When there are cavities all around the ring, can we meet the harmonic number jump condition everywhere?
  - The answer is no
  - But how good can we do?

Basic Equations

- We have a number of cavities at position θ_k in the ring, with maximum energy gain V_k, frequency f_k, and phase ϕ_k.
- The revolution time as a function of energy is given by $T(E)$
- We can write equations of motion for time t and energy E

$$\frac{dt}{d\theta} = \frac{T(E)}{2\pi} \qquad \frac{dE}{d\theta} = V(\theta)\cos\bigl(2\pi f(\theta)t + \phi(\theta)\bigr)$$

$$V(\theta) = \sum_k V_k \delta_{2\pi}(\theta - \theta_k) \qquad \delta_{2\pi}(\theta) = \sum_m \delta\bigl(\theta - 2\pi m\bigr)$$

$$f(\theta_k) = f_k \qquad \phi(\theta_k) = \phi_k$$

Single Cavity

- Single cavity located at $\theta_0 = \pi$
- We want to arrive at the cavity with the same phase each time

$$E(2\pi k) = E(0) + kV_0 \cos\phi_0 = E(0) + k\Delta E$$

$$t(2\pi k + \pi) = \sum_k T\bigl(E(0) + k\Delta E\bigr) - \frac{1}{2}T\bigl(E(0)\bigr)$$

- To have the same arrival phase each time

$$fT\bigl(E(0) + k\Delta E\bigr) = h_k$$

  h_k an integer
- $T(E)$ linear in E: $T(E) = T\bigl(E(0)\bigr) + \Delta T[E - E(0)]/\Delta E$
- Thus $f\bigl[T\bigl(E(0)\bigr) + k\Delta T\bigr] = h_k$
- Harmonic number jump condition: $f\Delta T = m$, m an integer

Cavities Everywhere

- Energy gain distributed uniformly

$$E(\theta) = E(0) + \Delta E \frac{\theta}{2\pi}$$

- Assuming time depends linearly on energy,

$$t(\theta) = t(0) + T\bigl(E(0)\bigr)\frac{\theta}{2\pi} + \frac{\Delta T}{2}\left(\frac{\theta}{2\pi}\right)^2$$

- We want to come back to the same phase on each turn

$$f(\theta)[t(\theta+2\pi) - t(\theta)] = f(\theta)\left[T\bigl(E(0)\bigr) + \frac{\Delta T}{2} + \Delta T\frac{\theta}{2\pi}\right] = h(\theta)$$

$h(\theta)$ is an integer

Cavities Everywhere
Mathematical Contradictions

- Can compute the cavity frequencies:

$$f(\theta) = \frac{h(\theta)}{T\bigl(E(0)\bigr) + \frac{\Delta T}{2} + \Delta T \frac{\theta}{2\pi}}$$

- Harmonic number jump condition

$$h(\theta + 2\pi) - h(\theta) = f(\theta)\Delta T = m$$

- Combining these

$$\frac{m}{\Delta T} = \frac{h(\theta)}{T\bigl(E(0)\bigr) + \frac{\Delta T}{2} + \Delta T \frac{\theta}{2\pi}}$$

Left side constant, right side is not

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How Good Can We Do?
Same Frequencies

- First attempt: make all cavities have the same frequency f
- Harmonic number jump condition: $f\Delta T = m$
- Cavity at $\theta = 0$ is synchronized

$$f\left[T\big(E(0)\big) + \Delta T/2\right] = h$$

- Cavity at $\theta = \pi$, the phase difference (divided by 2π):

$$f[T\big(E(0)\big) + \Delta T] = h + m/2$$

For m odd, it is completely out of phase

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How Good Can We Do?
Get First Turn Right

- Adjust cavity frequencies to get first turn right: on $\theta \in (0, 2\pi)$,

$$f(\theta) = \frac{h(0)}{T\bigl(E(0)\bigr) + \frac{\Delta T}{2} + \Delta T \frac{\theta}{2\pi}}$$

- Now check harmonic number condition

$$h(\theta) = f(\theta)[t(\theta + 2\pi) - t(\theta)] =$$

$$h(0)\frac{T\bigl(E(0)\bigr) + \frac{\Delta T}{2} + \Delta T \frac{\theta}{2\pi}}{T\bigl(E(0)\bigr) + \frac{\Delta T}{2} + \Delta T \left(\frac{\theta}{2\pi} - \left\lfloor \frac{\theta}{2\pi} \right\rfloor\right)}$$

Would like this to be an integer, but it isn't for all θ

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How Good Can We Do? Get First Turn Right (cont.)

- Compute harmonic number jump condition

$$h(\theta+2\pi)-h(\theta)=\frac{h(0)\Delta T}{T\bigl(E(0)\bigr)+\frac{\Delta T}{2}+\Delta T\left(\frac{\theta}{2\pi}-\left\lfloor\frac{\theta}{2\pi}\right\rfloor\right)}$$

- Make this an integer when $\theta = 2\pi n + \pi$

$$m=\frac{h(0)\Delta T}{T\bigl(E(0)\bigr)+\Delta T}$$

- Now write jump in terms of $h(0)$ and m

$$h(\theta+2\pi)-h(\theta)=\frac{h(0)m}{h(0)+m\left(\frac{\theta}{2\pi}-\left\lfloor\frac{\theta}{2\pi}\right\rfloor-\frac{1}{2}\right)}$$

How Good Can We Do? Get First Turn Right (cont.)

- Worst case error in harmonic number difference looks like

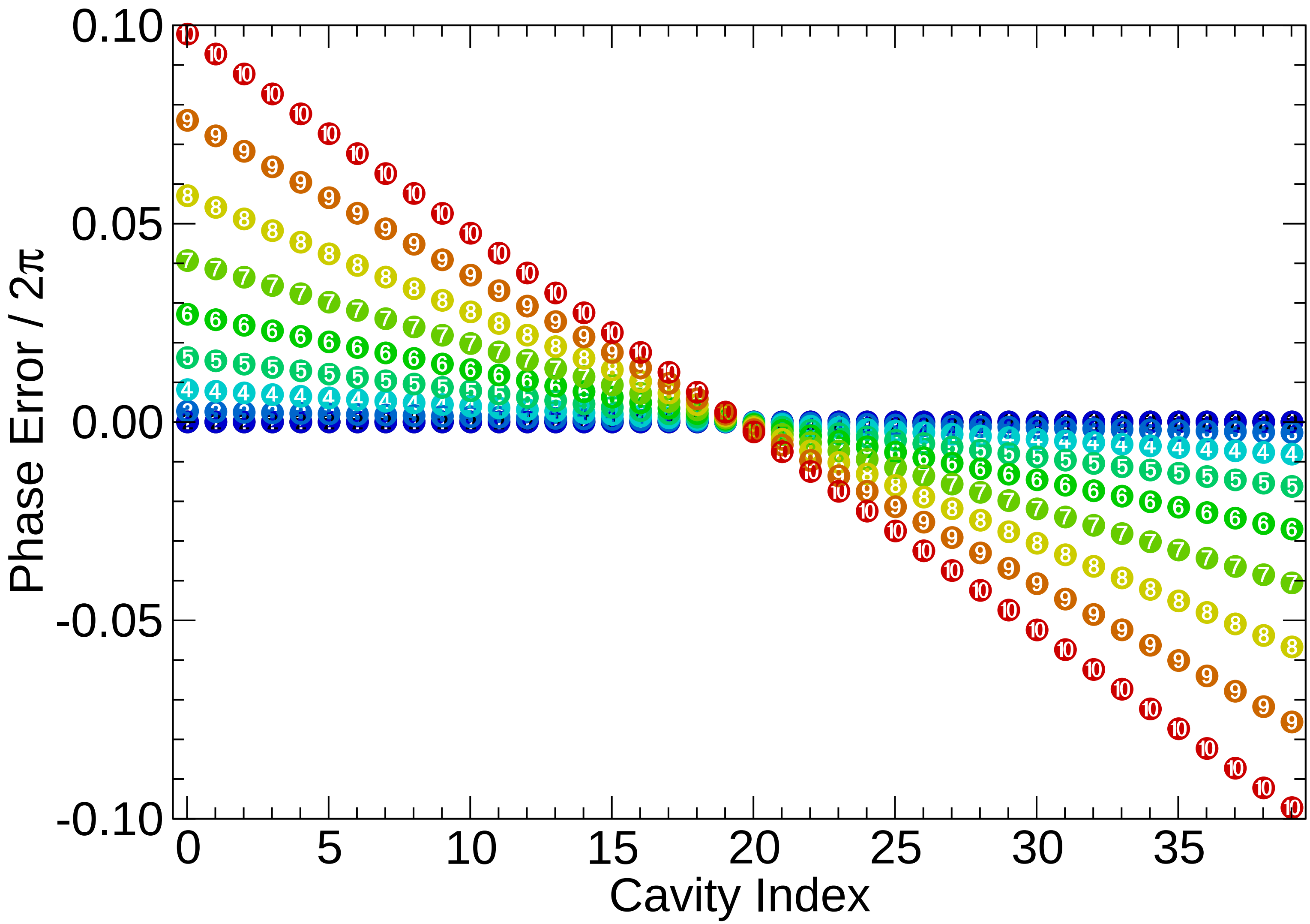
$$\frac{m^2}{2h(0) \pm m}$$

  - This is the **increase** in the **period** error on each turn
  - The phase error is the sum of the period errors
- After n turns, the worst phase error is thus

$$2\pi\frac{(n-1)(n-2)}{2}\frac{m^2}{2h(0) \pm m}$$

Approximately $\sqrt{h(0)}/m$ turns before some cavities start decelerating

Cavity Phases
Synchronized to First Turn



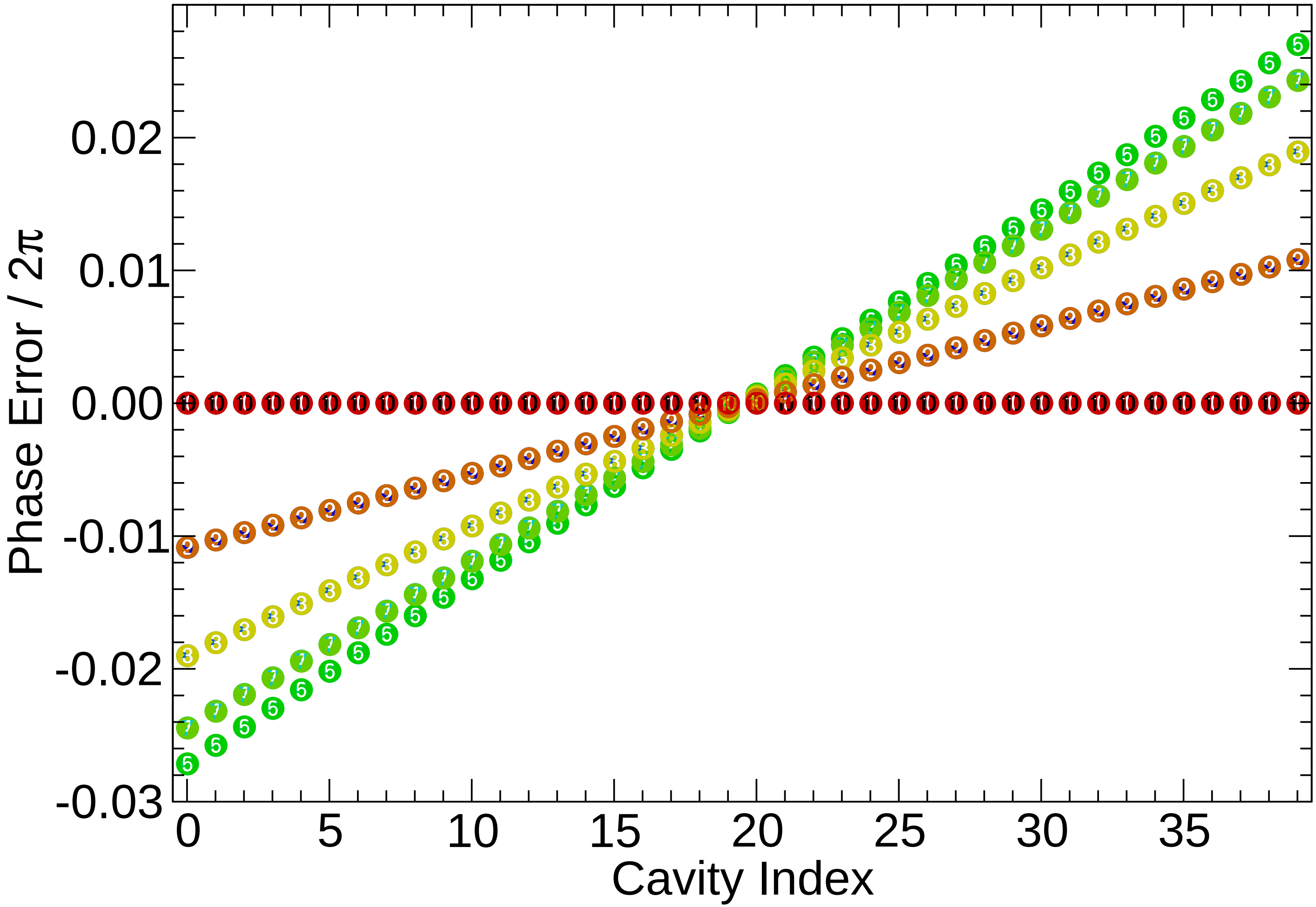
How Good Can We Do? Get Middle Turn Right

- Follow the same procedure, but make harmonic number jump correct for central turn
- After n turns, the worst phase error is thus

$$\pi \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \frac{m^2}{2h(0) \pm m}$$

  - About 1/4 of the phase slip compared to getting first turn right
  - Twice as many turns possible

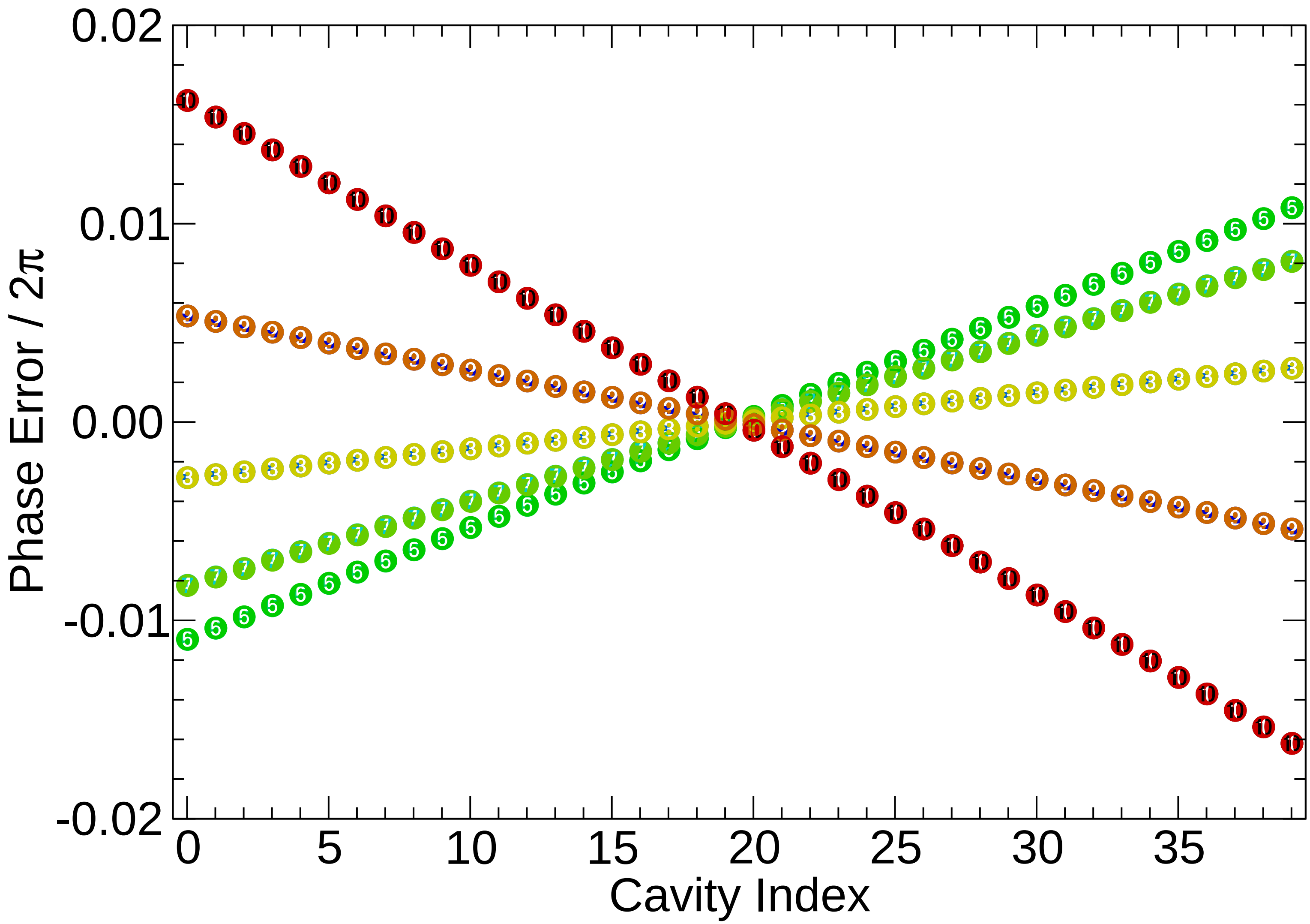
Cavity Phases
Synchronized to Middle Turn



How Good Can We Do?
More Ideas

- Could make average phase slip zero at each cavity
- Note that in all cases, average phase error is zero
  - On linear RF, average effect is small
  - On crest, unsynchronized turns gain less energy
    - Exploit to vary energy gain with time?

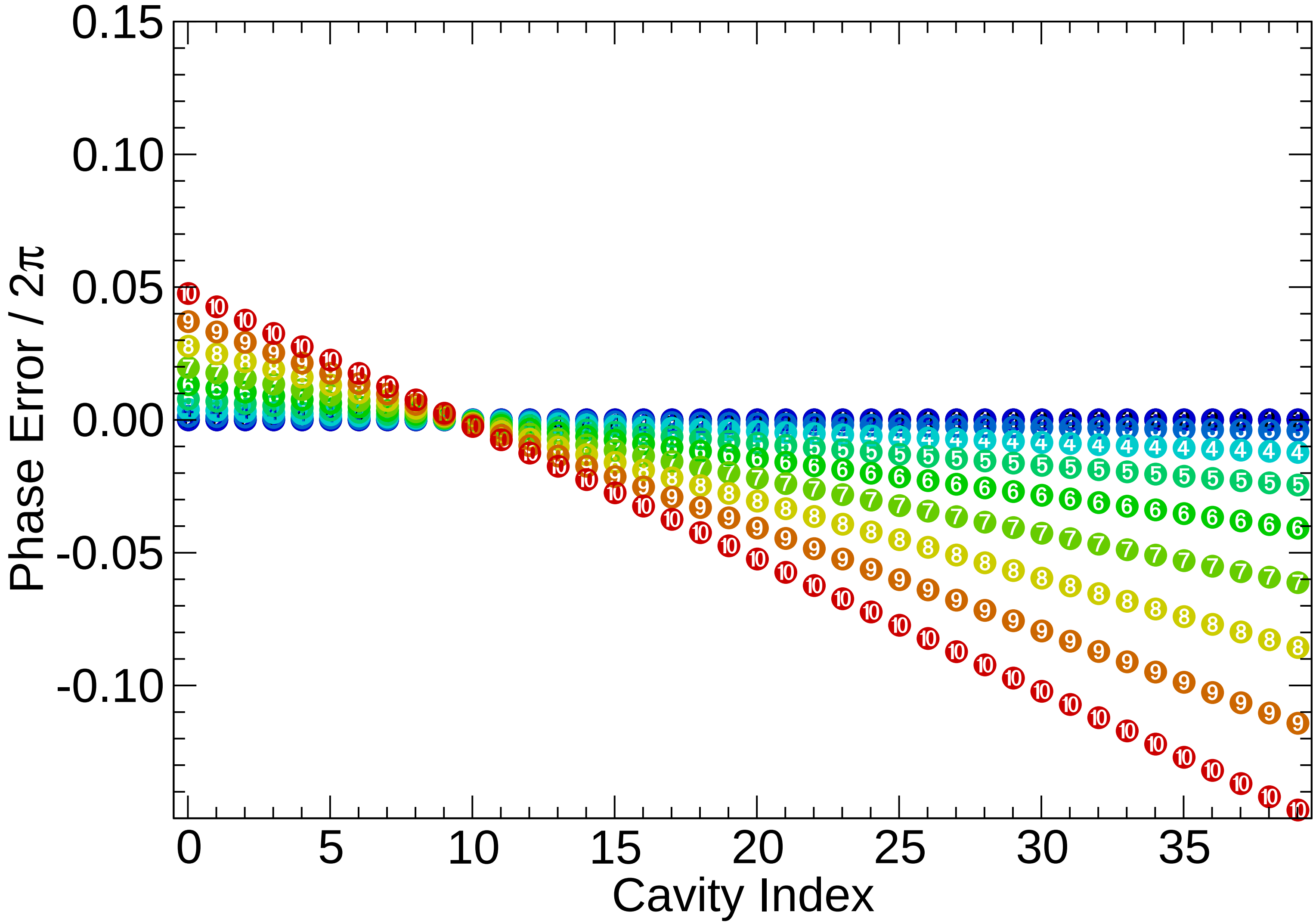
Cavity Phases
Synchronized to Middle Turn, Average Zero



Exploiting This Effect

- Use this to construct desired energy vs. time profile?
- Have three knobs
  - Which cavity is synchronized
  - Which turn has “zero” phase error
  - Which turn meets the harmonic number condition
- These control, respectively
  - The size of the average phase error on each turn
  - At a given cavity, the phase averaged over all turns
  - What phase you spend more time at, how phase varies during acceleration
- Could also use different phases or frequencies than these

Cavity Phases
Synchronized to Middle Turn, Average Zero



Application to Non-Muon Rings

- Other rings don't need the ring filled with cavities
- However, could intentionally add additional cavities near, but not right next to, first cavity
  - May allow control of energy gain with turn number
  - If cavities too far apart, will not work for many turns

What is Lacking: the To-Do List

- No variation of energy gain with arrival time
- Time of flight must vary nonlinearly with energy
- No computation of RF bucket
- What happens with a long bunch train?

Conclusions

- When the ring is filled with cavities, the harmonic number jump conditionn cannot be satisfied, even when time of flight depends linearly on energy
- By choosing the cavity phases and frequencies correctly, one can do very well
- One may even be able to exploit this to vary the energy gain with turn number
  - One may even be able to exploit this for a ring needing very few cavities
- This is a highly simplified calculation at this point, and there is more work to be done.