Longitudinal Phase Space Distortion in FFAGs

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Outline

- Types of Machines for Acceleration
- What is an FFAG?
- Longitudinal dynamics in one type of FFAG
- Calculation of longitudinal phase space distortion
    - “Emittance” growth
    - Ellipse distortion

Types of Machines for Acceleration

- Linac
  - Accelerates extremely rapidly
  - All sorts of other good properties
  - Only a single pass through the RF: expensive
- Recirculating linac (e.g., CEBAF)
  - Make multiple passes through the same linac: save money
  - Still accelerate very rapidly
  - Need a separate arc for each pass
  - Number of passes limited (typically 5 or so)
    - Beam overlap in successive passes
    - Complexity of switchyard
    - Separate arcs cost money

Types of Machines for Acceleration

- Synchrotron
  - Make a huge number of passes through the same cavity
  - Increase magnet fields in proportion to the reference momentum
    - Thus, the beamline effectively deals with only a small energy range (the energy spread in the beam)
    - Acceleration rate limited by how fast you can ramp magnets
    - RF frequency also must change for nonrelativistic particles. This may also limit the acceleration rate.
- Cyclotron
  - Magnetic fields don't vary: can accelerate quickly
  - Isochronous, so RF frequency doesn't vary
  - Many passes through same cavity
  - Orbit position changes as you accelerate
  - Weak focusing: the machine is HUGE

What is an FFAG?

- FFAG stands for **F**ixed **F**ield **A**lternating **G**radient
- Make many passes through your RF cavities: reduce costs
- A single beamline for all energies (a factor of 2 or more in momentum)
- Magnetic fields don't vary as you accelerate: allows for rapid acceleration
  - These machines are probably not interesting if you don't want to accelerate rapidly
- Alternating gradient focusing gives stronger focusing
  - Beam size smaller
  - Closed orbits at different energies closer together (smaller dispersion)

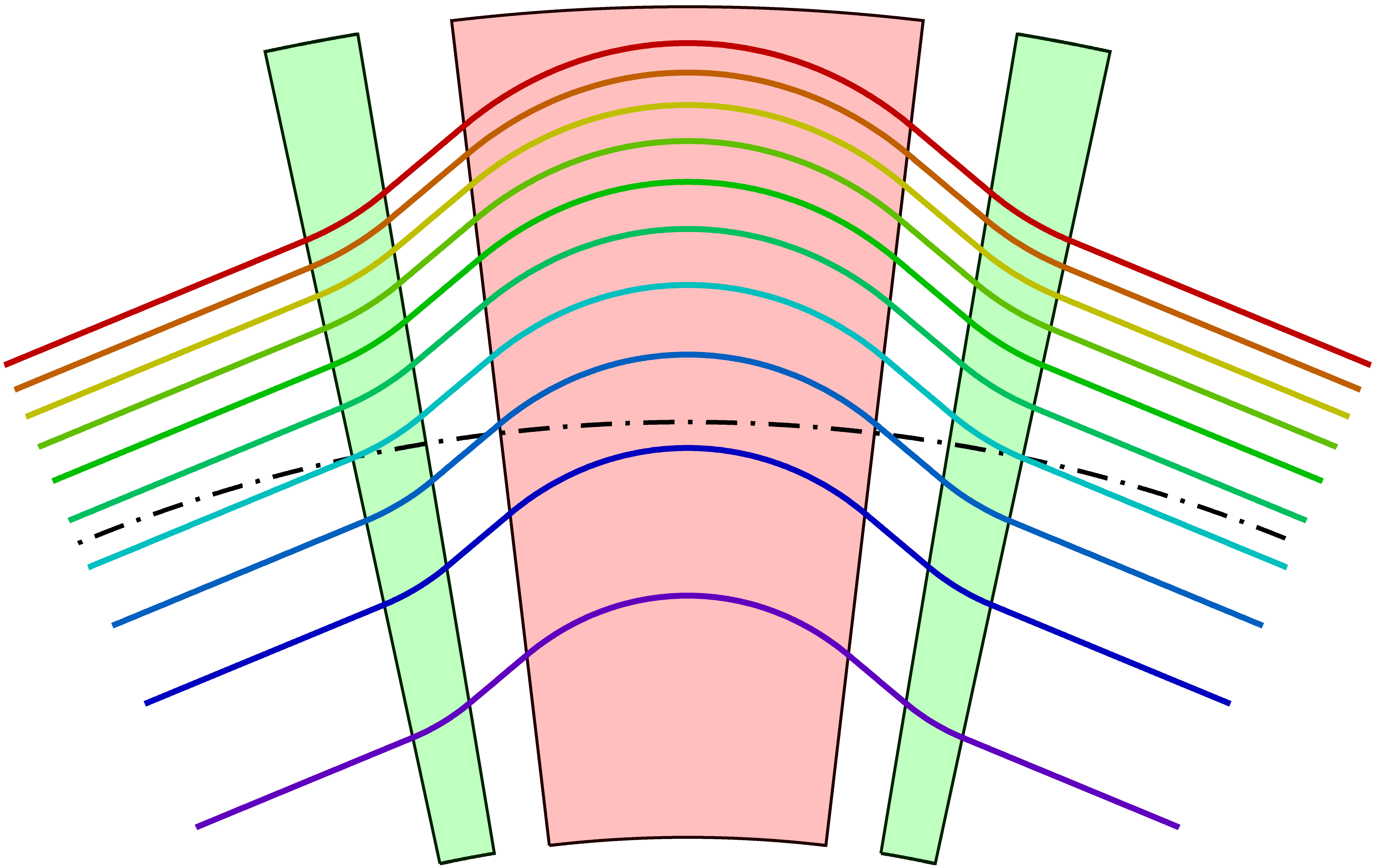
Scaling FFAGs

- FFAGs were originally developed in the 50s.
- They created a machine with exactly zero chromaticity over a large energy range
  - The trick: make the magnetic field in the midplane be of the form $B_y(\theta)(r/r_0)^k$, where θ and r are cylindrical coordinates with respect to the center of the machine.
  - Exercise for the listener:
    - Write out the Hamiltonian in r and θ
    - Use Maxwell's equations to construct vector potentials as an infinite series
    - Demonstrate that by applying a linear transform which only depends on energy, the motion becomes independent of energy!

Scaling FFAGs (cont.)

- Tunes and momentum compaction (defined carefully) are independent of energy
- Phase space identical for all energies except for energy-dependent linear transform
- Just as in synchrotron, find a good operating point and you stay there
- Get a kind of adiabatic invariance of phase space
- Closed orbits are geometrically similar
- KEK has built and operated these things recently!

Scaling FFAG: Closed Orbits



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The KEK PoP FFAG

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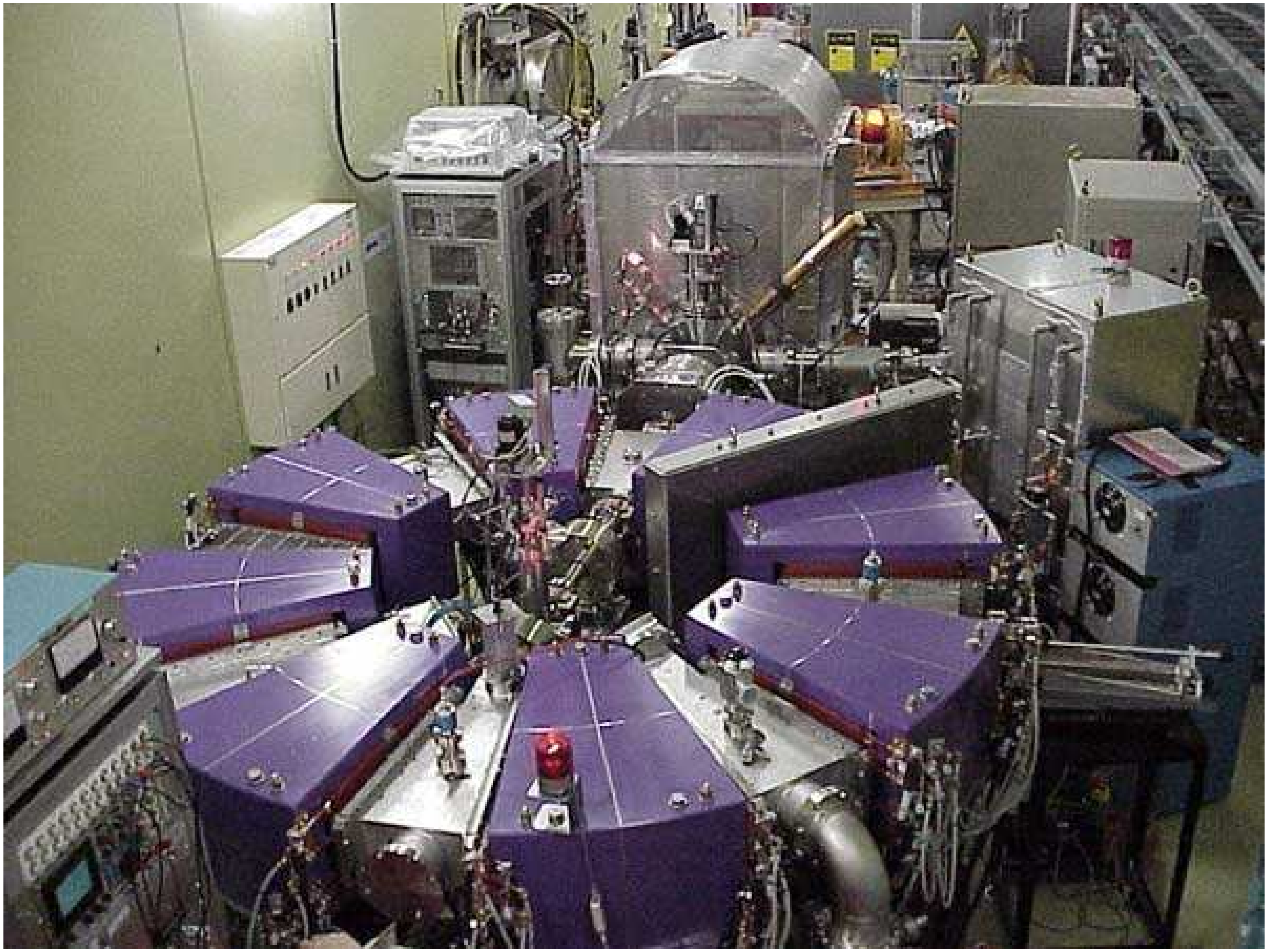


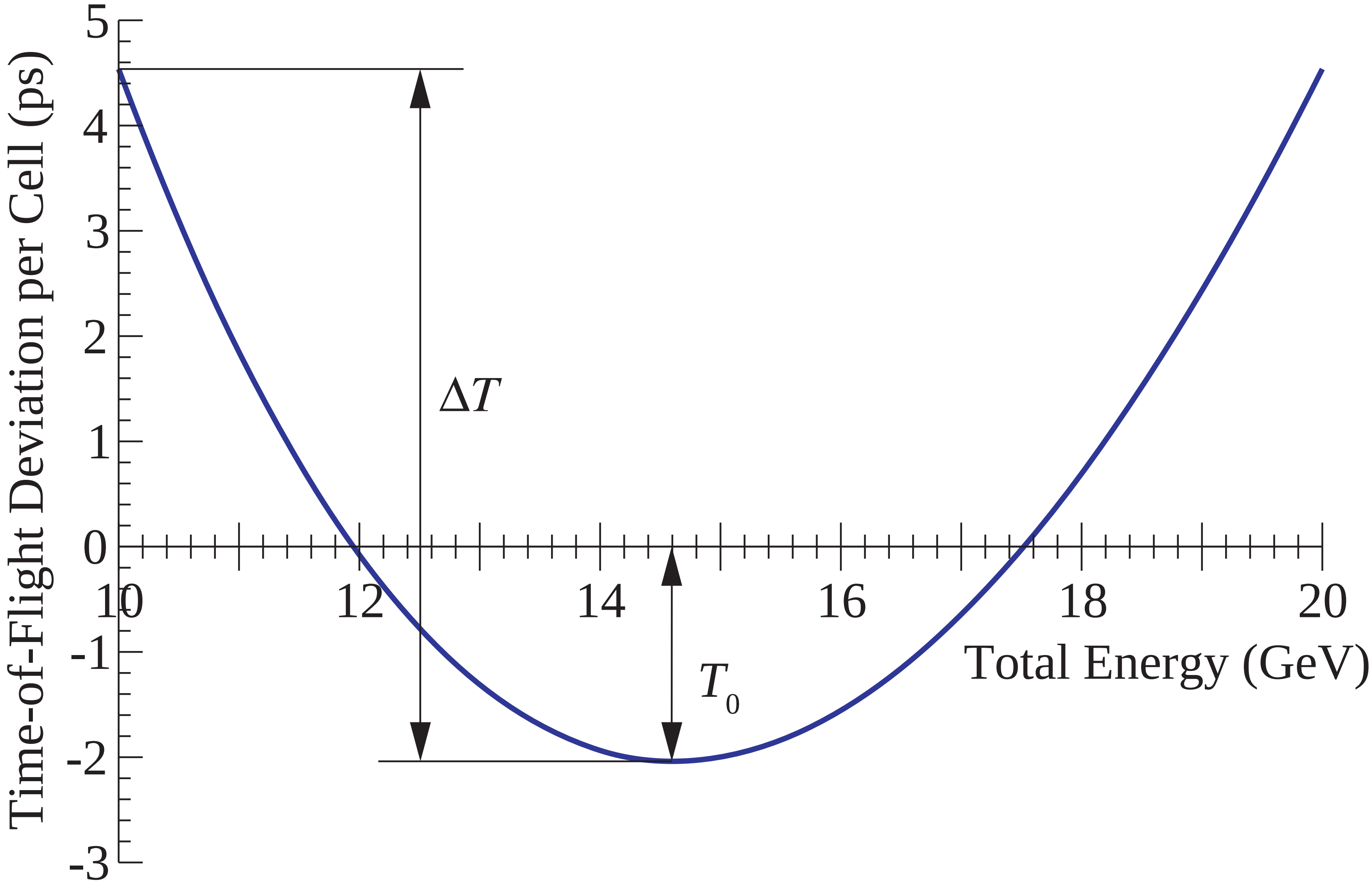
Non-Scaling FFAGs

- Eliminate the scaling properties of the "scaling" FFAG
- Benefits:
  - Smaller magnet apertures
  - More isochronous: don't vary RF frequency as much
  - Can use linear magnets: better dynamic aperture at fixed energy
- Problem: no longer have zero chromaticity. Resonances!
  - Have extremely high degree of symmetry
    - All cells are identical
    - Cells are short (doublet, triplet), with tune below 0.5
  - Linear magnets can minimize driving of these resonances
  - Rapid acceleration means we don't have much of an effect from resonances

Time of Flight

- FFAGs are not isochronous
- Can vary RF frequency to compensate
- If you're trying to accelerate very quickly (muons!), can't vary RF frequency fast enough
  - You must accelerate fast enough to avoid getting out-of-sync with the RF
- Minimize time-of-flight range: time-of-flight nearly parabolic

Time-of-Flight vs. Energy



Goal of the Calculation

- Beam is being transferred from one machine to another
- We can straightforwardly manipulate the phase space ellipse linearly
- Any nonlinear distortions to the ellipse are difficult to reverse at best
- These nonlinear distortions cause problems
  - Beams sizes/energy spreads/whatever are larger than desired
  - A correction scheme may not be able to fix it
- Want to quantify these distortions
- Use the results to specify a machine design

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Longitudinal Dynamics

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- Want to study the longitudinal dynamics in the system with
  - Parabolic time-of-flight as a function of energy
  - RF smoothly distributed around the ring
- In particular, we want to characterize the longitudinal phase space distortion
- Time of flight is approximately a parabolic function of energy

$$\frac{d\tau}{ds} = \Delta T \left(\frac{2E - E_i - E_f}{\Delta E}\right)^2 - T_0,$$

- Energy gain from RF

$$\frac{dE}{ds} = V\cos(\omega\tau),$$

Normalized Variables

- Change of variables

$$x = \omega\tau \qquad p = \frac{E - E_i}{\Delta E} \qquad u = \frac{s}{\omega\Delta T}$$

  - Accelerate from $p = 0$ to $p = 1$

- New equations of motion

$$\frac{dx}{du} = (2p-1)^2 - b \qquad \frac{dp}{du} = a\cos x \qquad a = \frac{V}{\omega\Delta T\Delta E} \qquad b = \frac{T_0}{\Delta T}$$

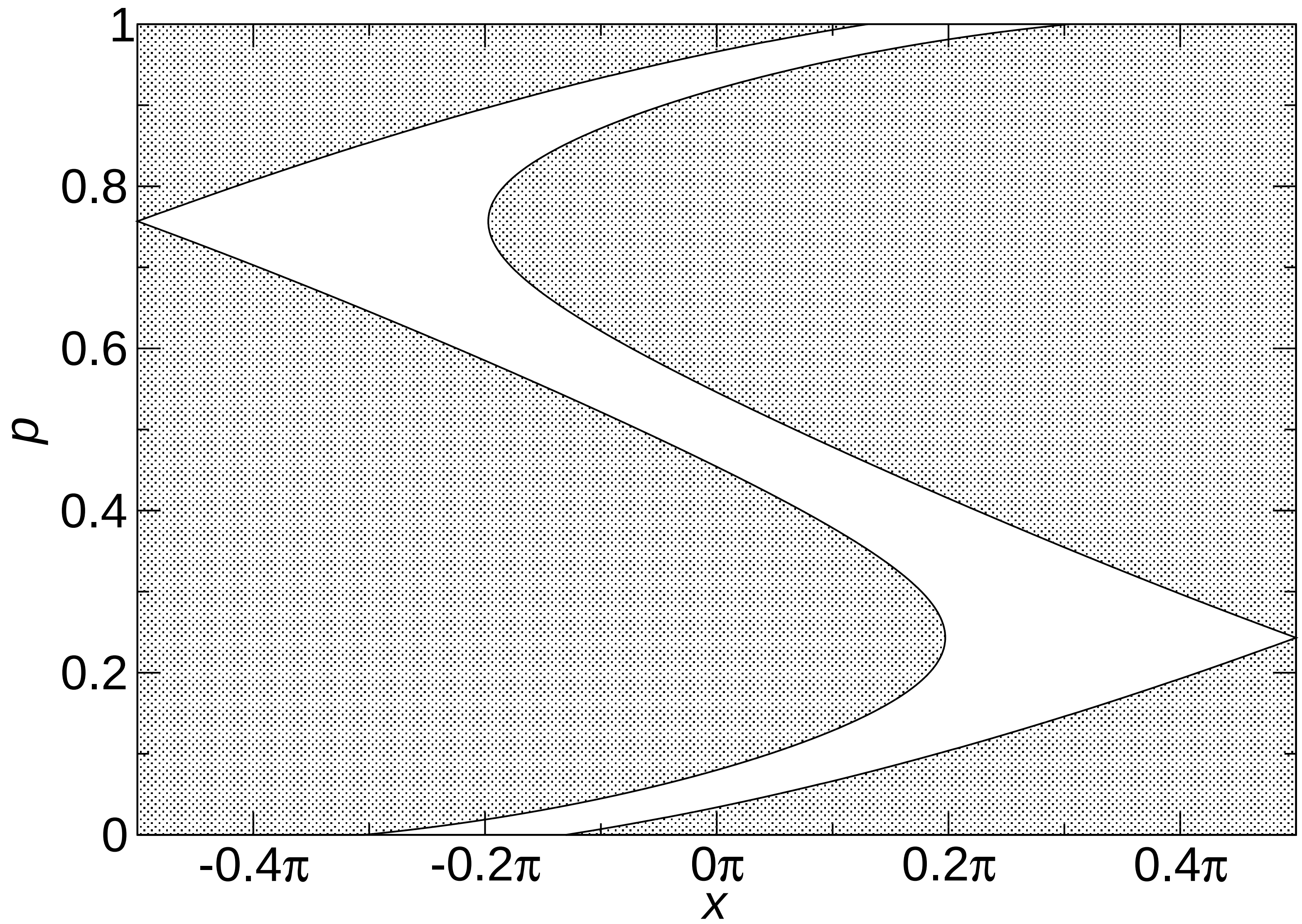
- Hamiltonian

$$\frac{1}{6}(2p-1)^3 - \frac{b}{2}(2p-1) - a\sin x$$

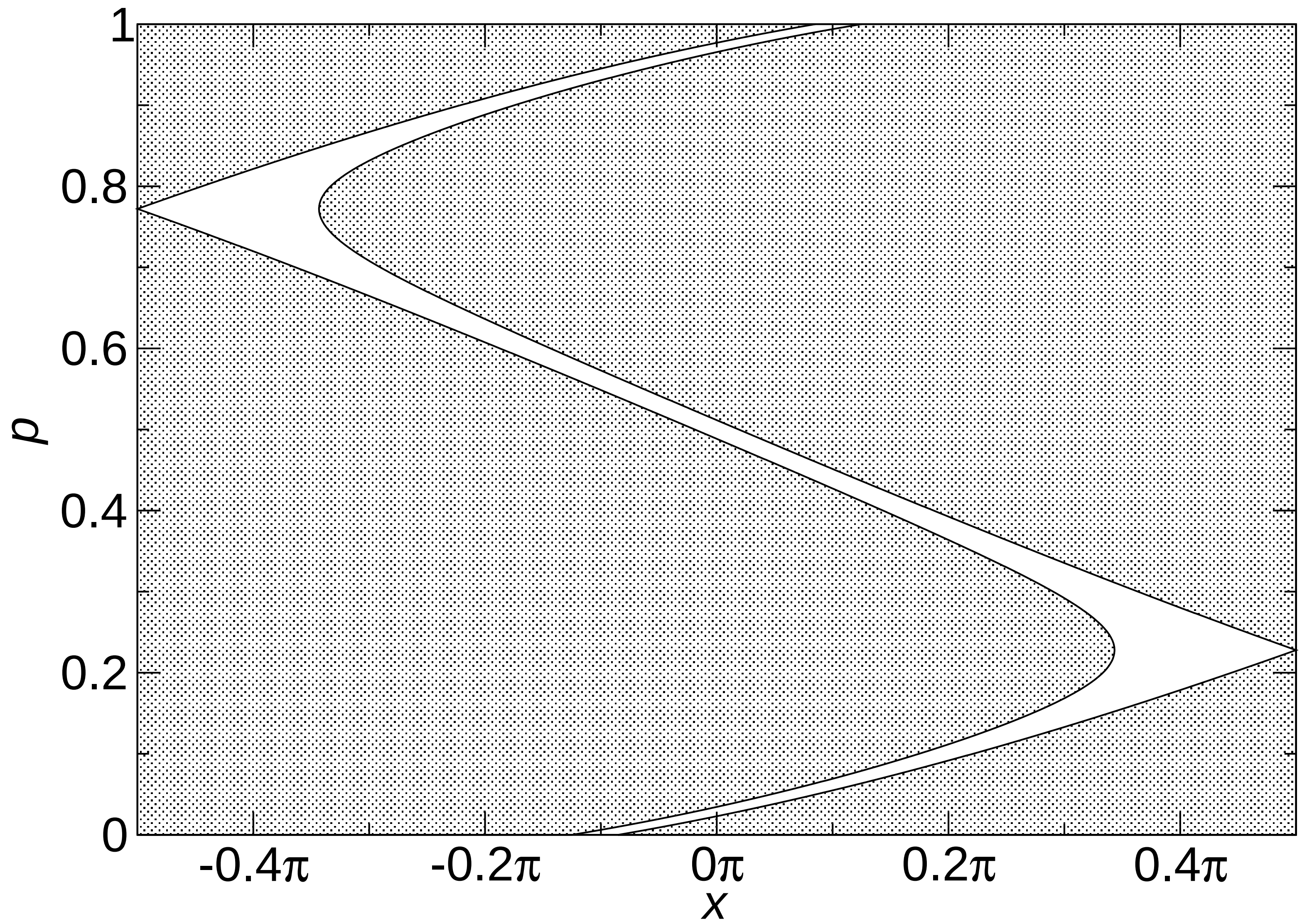
Parameter Regimes

- Dynamics depend completely on two dimensionless parameters, a and b
- To pass particles through from $p = 0$ to $p = 1$, require $a > b^{3/2}/3$
- For central particle to cross $p = 0$ and $p = 1$, require $a > |1/6 - b/2|$
- Small a, smaller phase space region for bunch
- Requirements together lead to minimum a of $1/24$
  - Smaller a gives more emittance growth
- Based on design requirements (emittance, allowed emittance growth, etc.), determine a and b

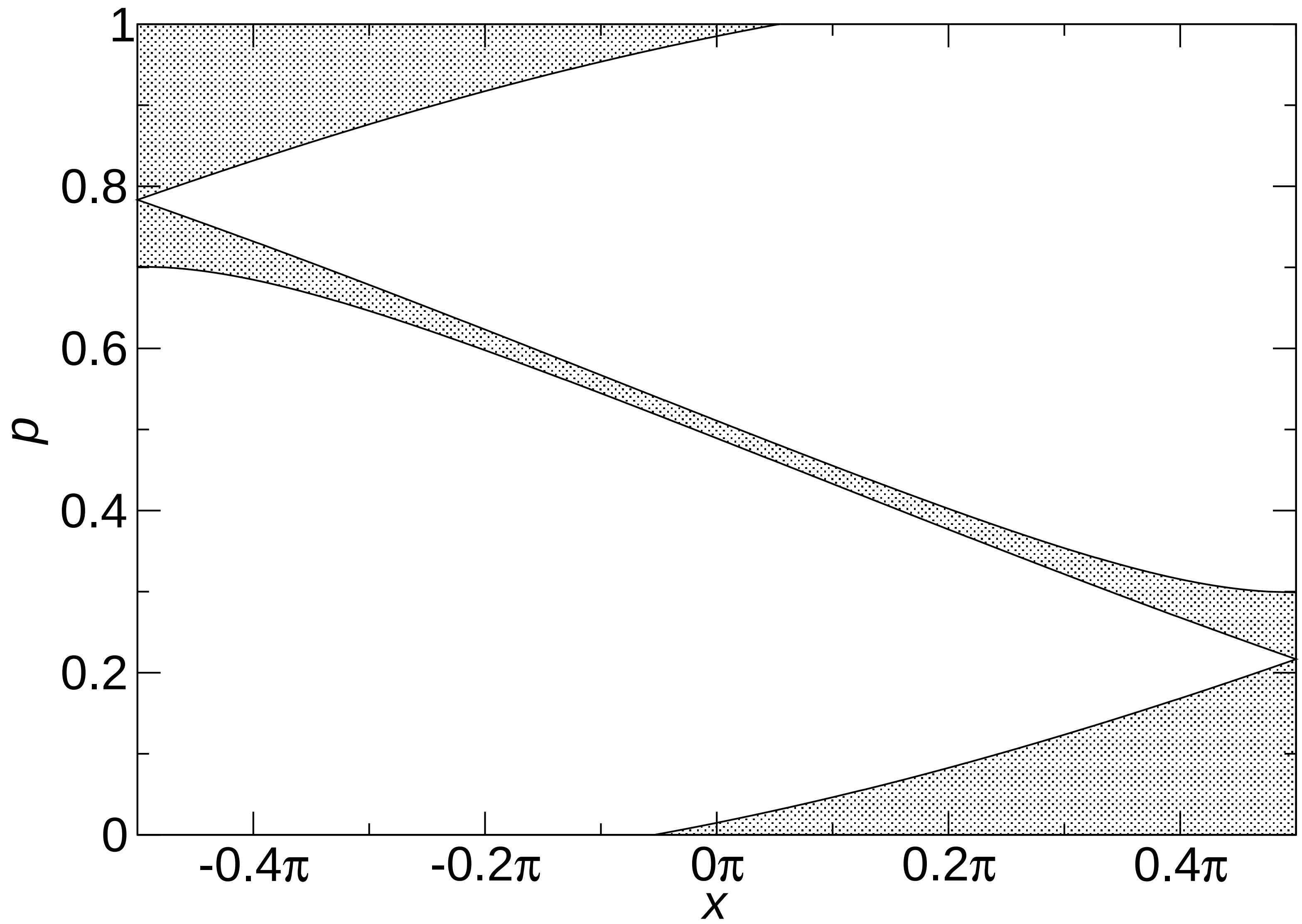
Particles Passing Through



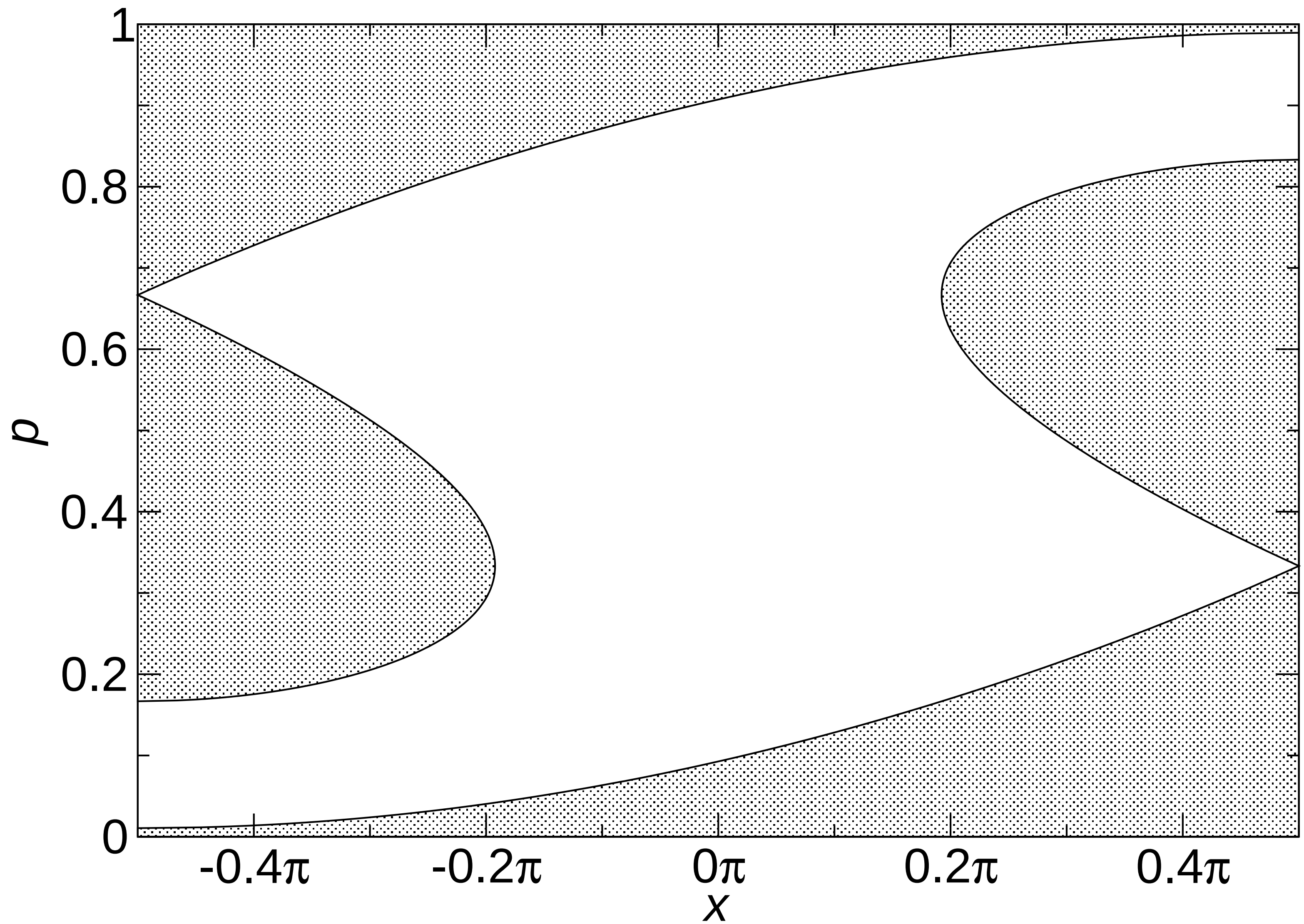
Particles Barely Pass



Particles Can’t Pass



Central Particle Doesn't Make It



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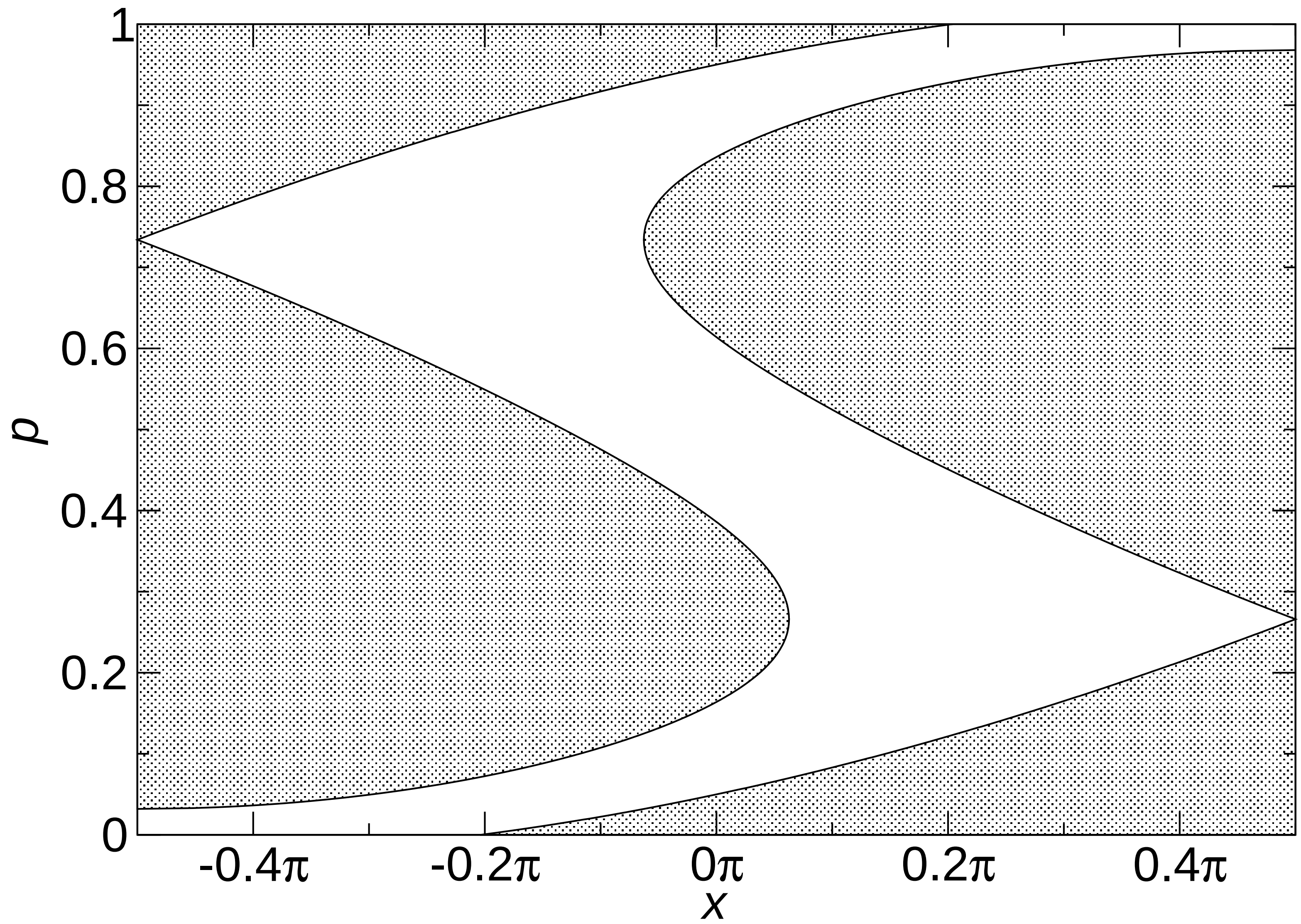
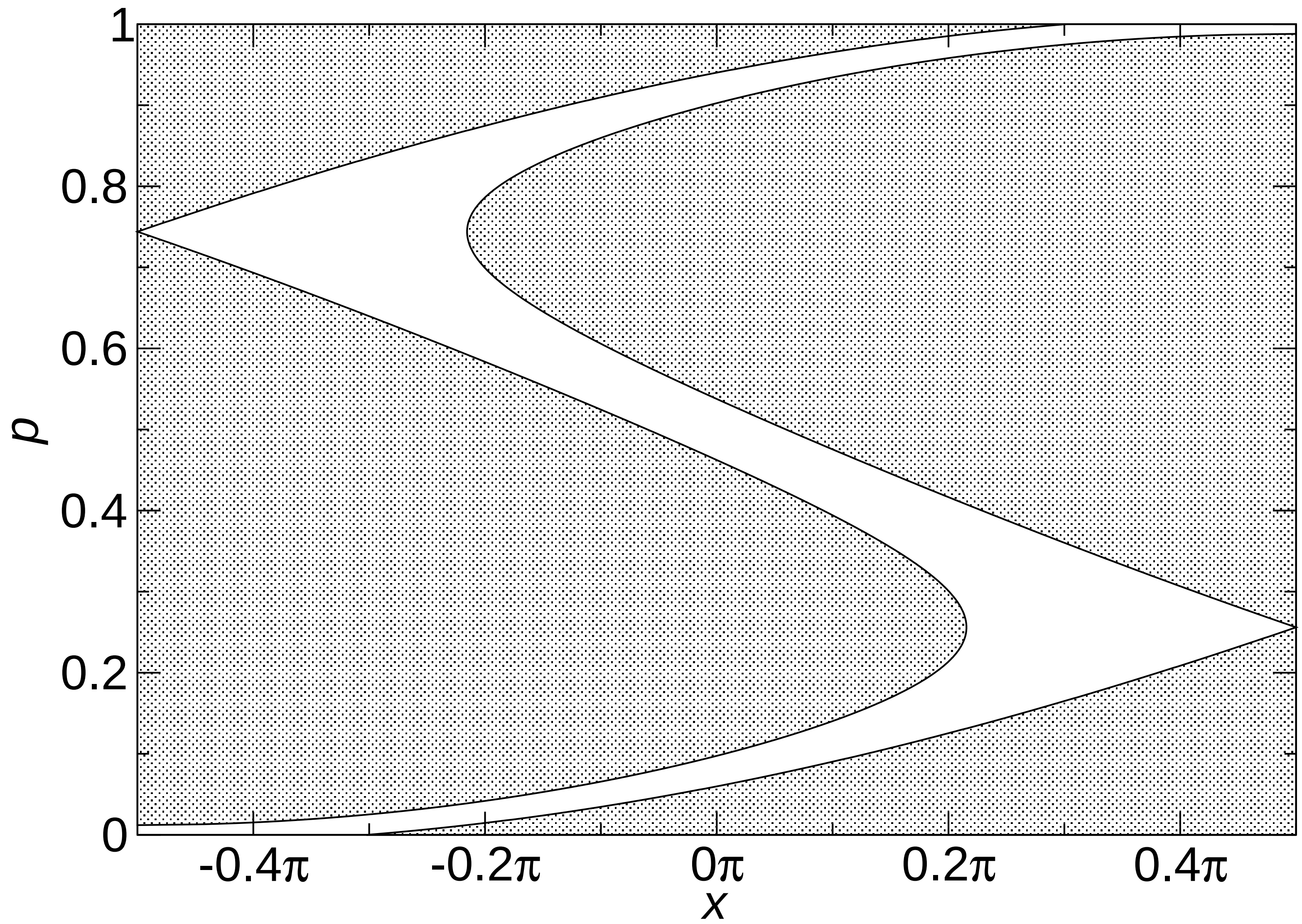
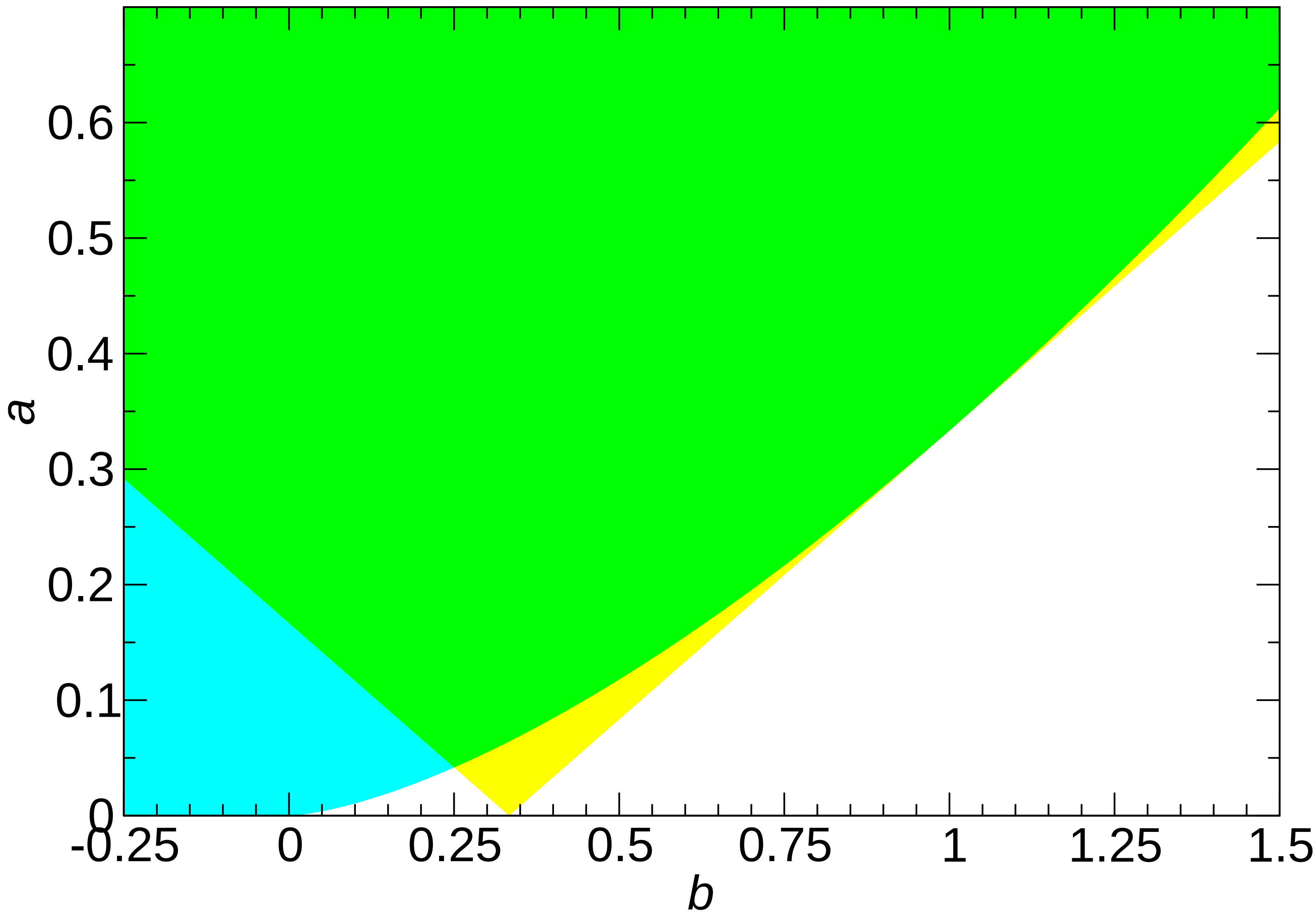


Lower a



Allowed Region of Parameter Space



The Dragt-Finn Factorization

- A general symplectic map can be described by a “Dragt-Finn Factorization”:

$$e^{-:g_1:}\dots e^{:f_5:}e^{:f_4:}e^{:f_3:}e^{:f_2:}e^{:f_1:}$$

  - I won’t go into what precisely this means…
- f_n is a nth-order homogeneous polynomial in the phase space variables
- f_1 describes the final reference point, g_1 the initial reference point
- f_2 is the linear part of the map
- The rest are nonlinear

Computing the Dragt-Finn Factorization

- Start with one solution of the equations of motion $\boldsymbol{z}_0(u)$
- Transform phase space variables $\boldsymbol{z}$ to $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{z}_0(u)$
- The Hamiltonian can be written as $h_2 + h_3 + h_4 + h_5 + \cdots$, where h_n is a homogeneous nth order polynomial in $\boldsymbol{w}$
- To compute $e^{:f_2:} = M$, M is a matrix,

$$\frac{dM}{du} = JH_2M \qquad\qquad h_2 = \frac{1}{2}\boldsymbol{w}^T H_2 \boldsymbol{w}$$

  and J is the symplectic metric matrix
- To compute f_3,

$$\frac{df_3}{du} = -e^{:f_2:}h_3$$

- Higher order terms have more complicated equations

Symmetries of this Problem

- The problem is invariant under the transformation $x \longrightarrow -x$, $p \longrightarrow 1 - p$
- The reference orbit we use passes through $(x, p) = (0, 1/2)$
- Thus, the map can be written as

$$\mathcal{M} = \exp\left(2{:}e^{-:f_2:}f_5{:}\right)\exp\left(2{:}e^{-:f_2:}f_3{:}\right) = e^{:g_3:}e^{:g_5:}$$

- The f_n are computed going from the middle to the edge of the phase space

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Compute Emittance Growth

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- Write g_n as

$$g_n = \sum_{k=0}^{n} g_{nk} x^{n-k} p^k$$

- Calculate the emittance using the second-order covariance matrix

$$\sqrt{\det\{\langle \boldsymbol{w}\boldsymbol{w}^T\rangle - \langle \boldsymbol{w}\rangle\langle \boldsymbol{w}\rangle^T\}}$$

- "Emittance" growth is "bad behavior" that we want to minimize
- To lowest order, the emittance growth for a circular distribution is

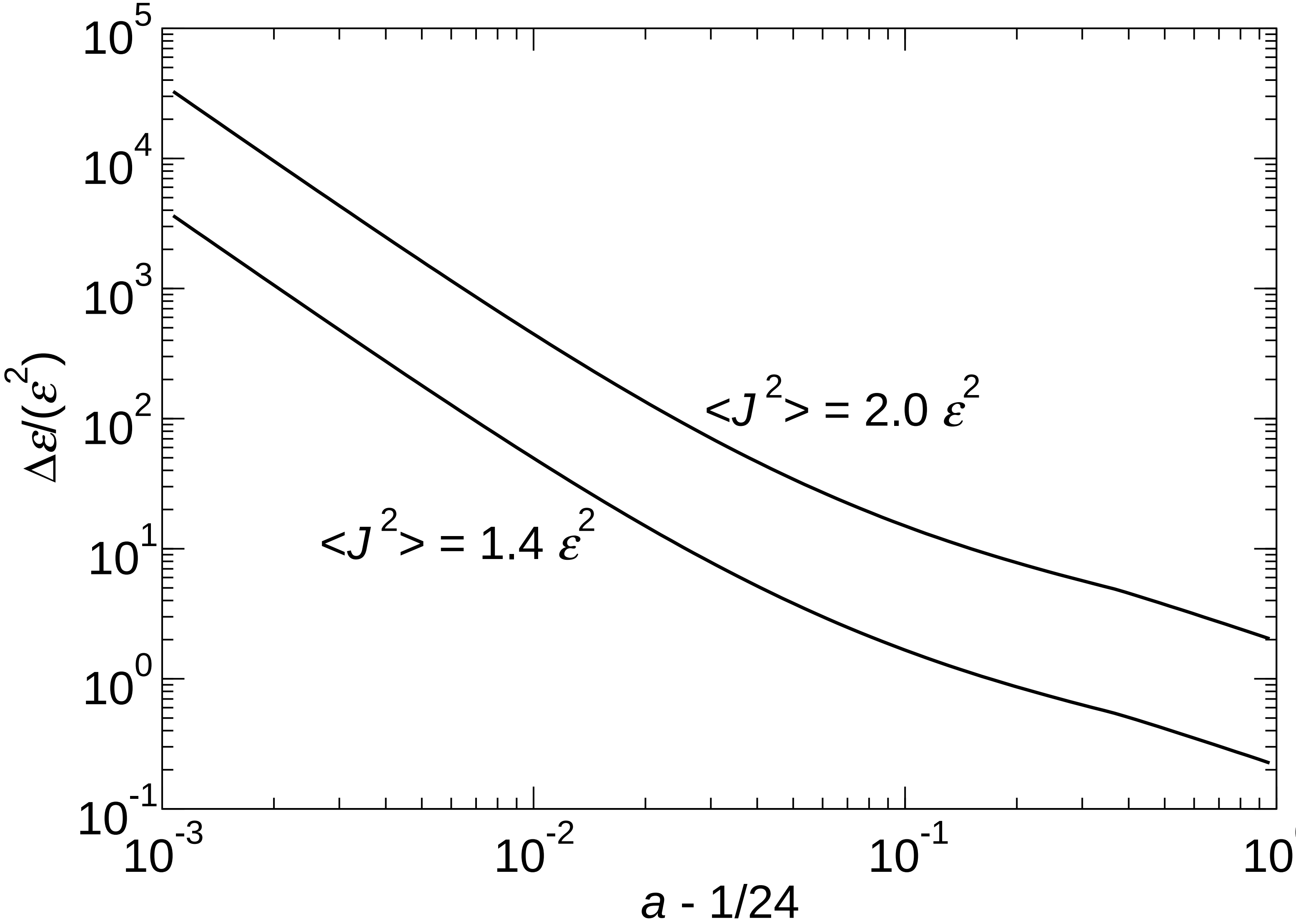
$$3\langle J^2\rangle(9g_{30}^2 - 6g_{30}g_{32} + 5g_{32}^2 + 9g_{33}^2 - 6g_{33}g_{31} + 5g_{31}^2)/4 \\ - \langle J\rangle^2[(3g_{30} + g_{32})^2 + (3g_{33} + g_{31})^2]/2$$

  - $\langle J\rangle = \epsilon$ is the emittance; $\langle J^2\rangle > \langle J\rangle^2$
  - This can be negative if $\langle J^2\rangle < (4/3)\langle J\rangle^2$ (equality for uniform)!

Computing Emittance Growth

- For given a and b, compute g_3
- Transform g_3 with a linear transform corresponding to the orientation of the incoming ellipse
  - Minimize emittance growth over that transform (two free parameters)
- Minimize the result with respect to b
- Have emittance growth as a function of a

Emittance Growth vs. a



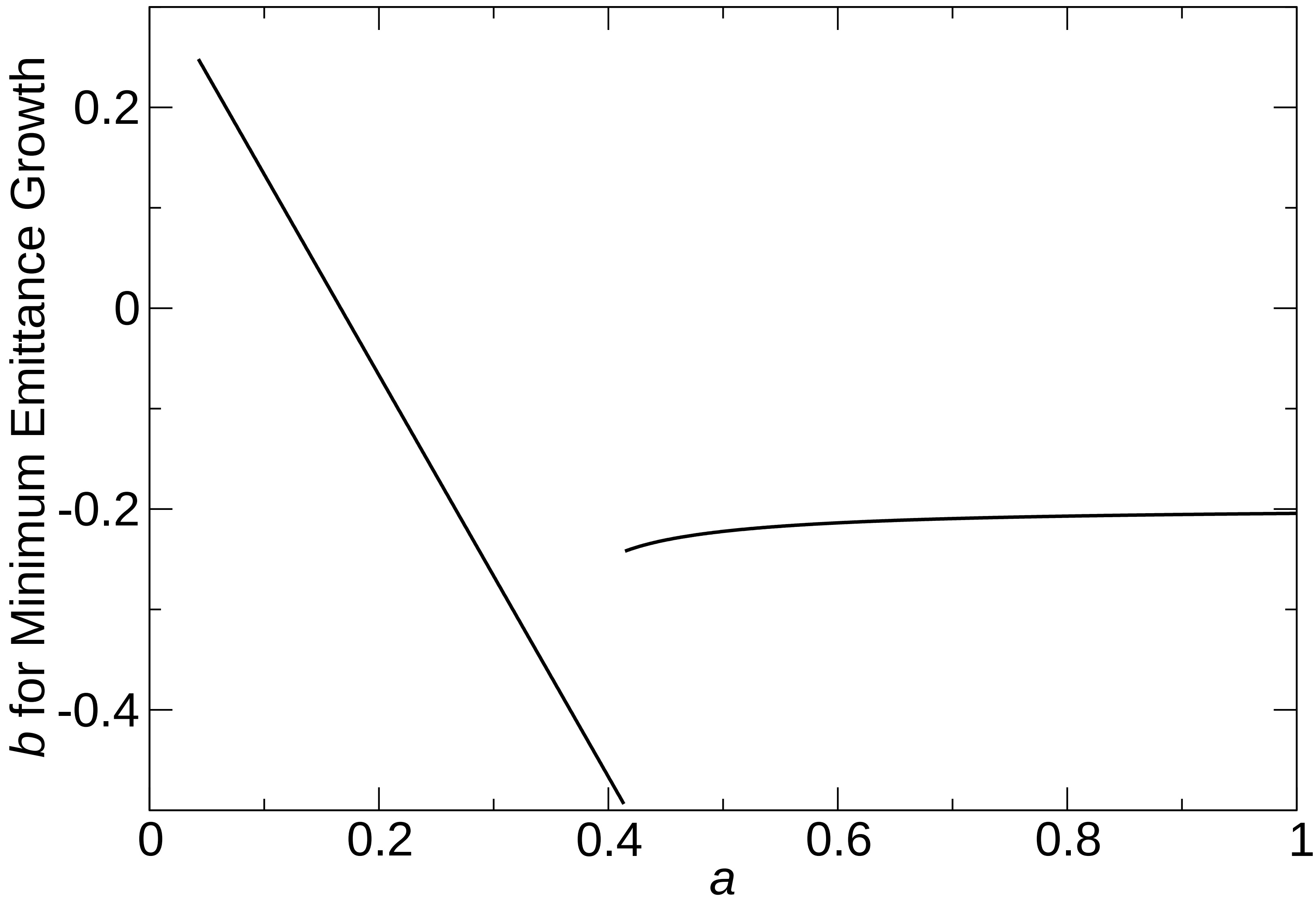
Emittance Growth Analysis

- For small a, $\Delta\epsilon/(\epsilon^2) \propto (a - 1/24)^{-2}$
  - Time-of-flight is proportional to $-\log(a - 1/24)$
  - Recall equation for linear matrix $M = e^{:f_2:}$: $dM/du = JH_2M$
  - Thus, matrix elements of M can be exponential in the time-of-flight, or proportional to $(a - 1/24)^{-1}$
  - Recall $df_3/du = -e^{:f_2:}h_3$
  - Thus f_3 can also be proportional to $(a - 1/24)^{-1}$
  - Emittance growth is quadratic in f_3, thus proportional to $(a - 1/24)^{-2}$

Emittance Growth Analysis (cont.)

- Emittance growth is smaller for smaller $\langle J^2\rangle/\epsilon^2$
- To use:
  - Compute emittance in normalized coordinates
  - Choose acceptable emittance growth
  - Find a which gives that emittance growth
- Optimal b is independent of $\langle J^2\rangle/\epsilon^2$
- For small a, optimal b is the minimum b
  - Can be negative!
- Optimal ellipse orientation is tilted, even though initial phase space trajectories are flat

Optimal b



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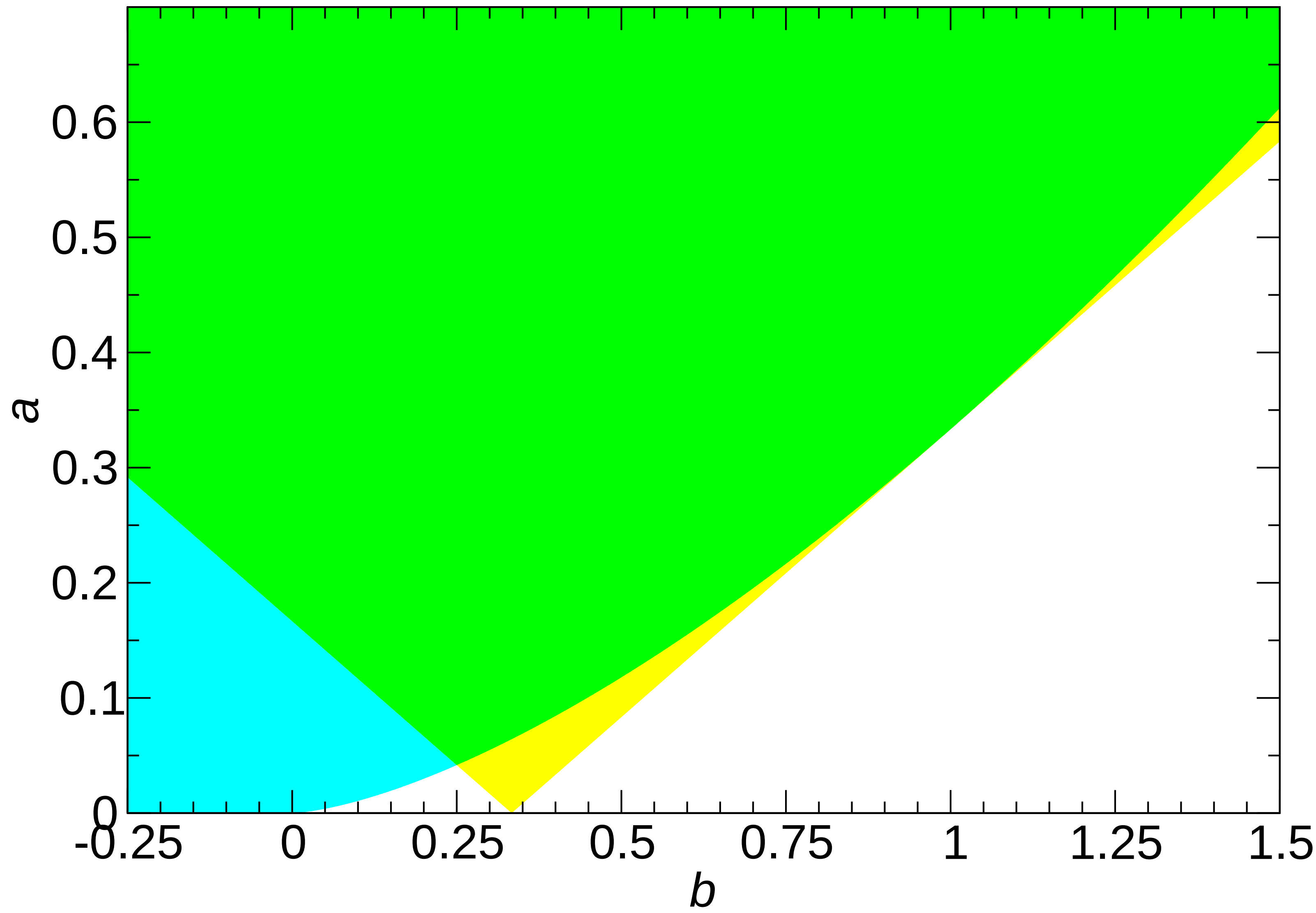
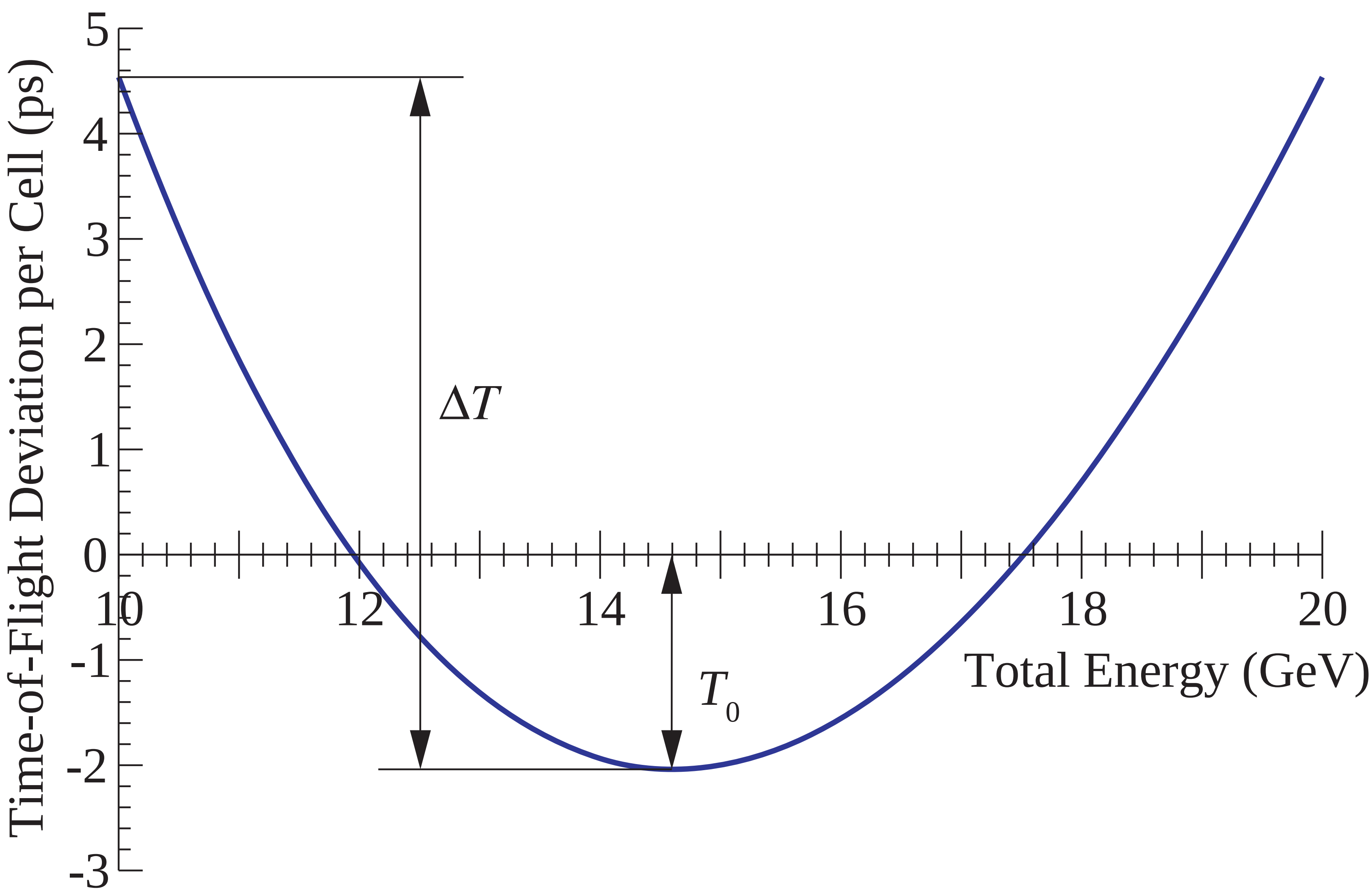
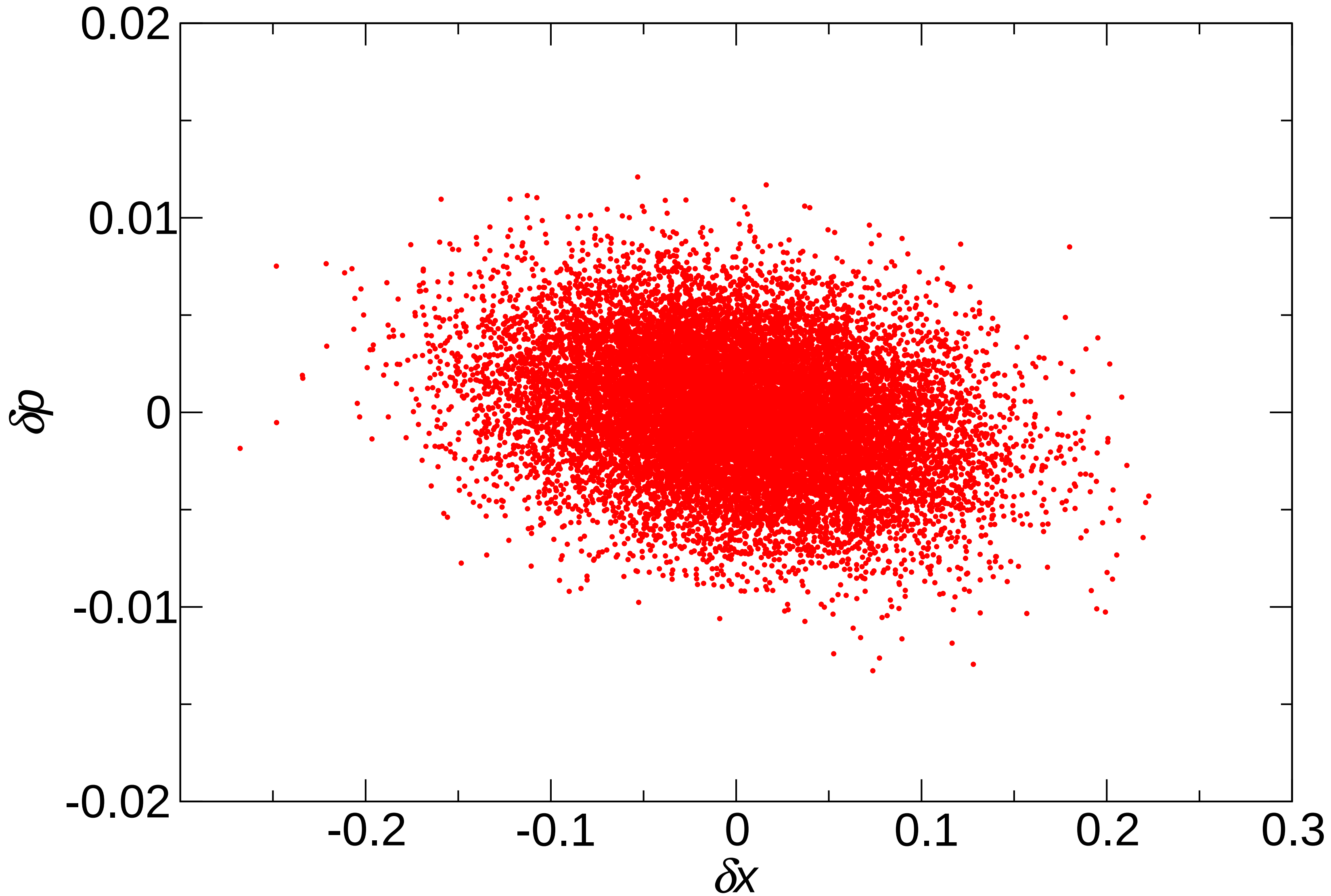


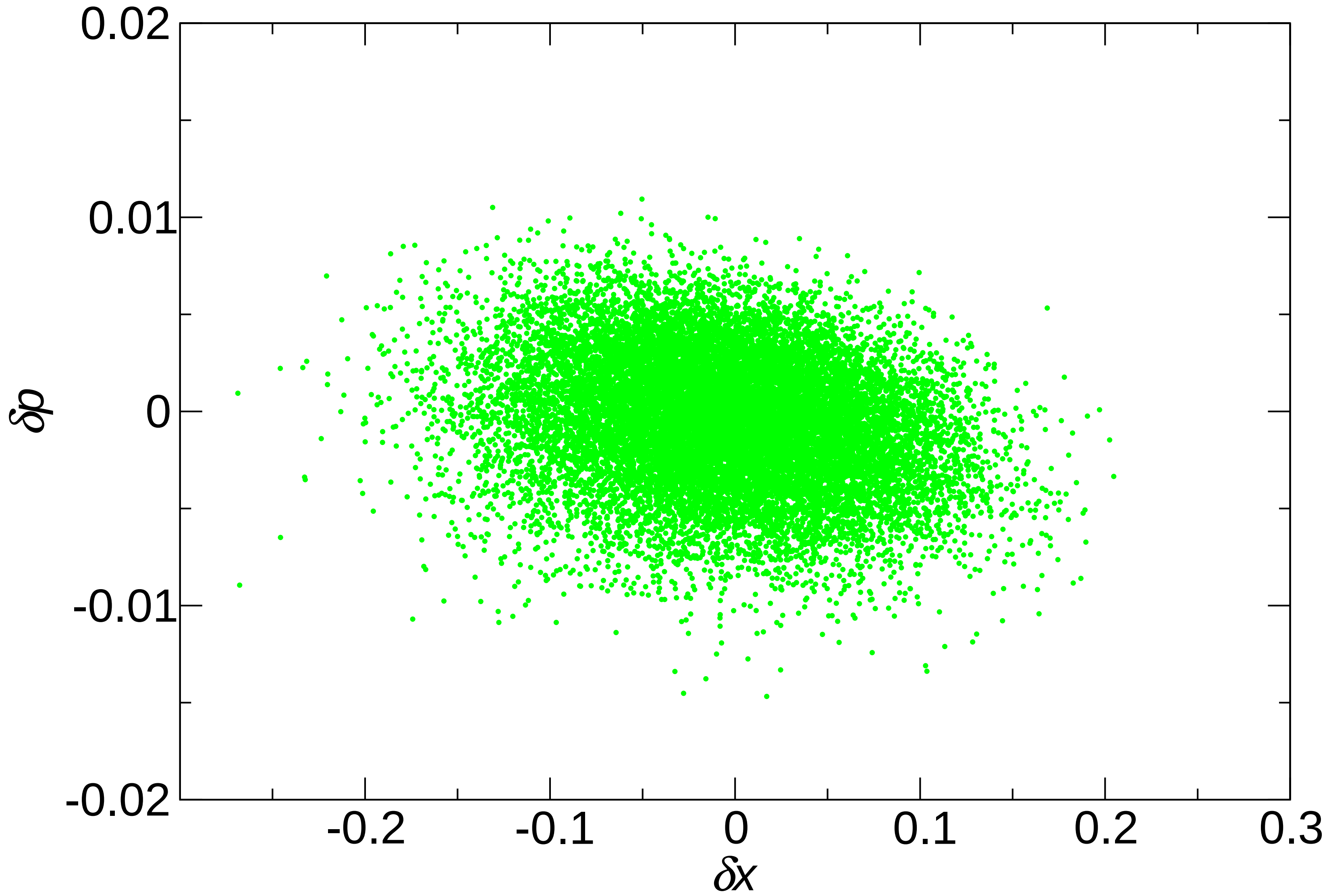
Time-of-Flight vs. Energy



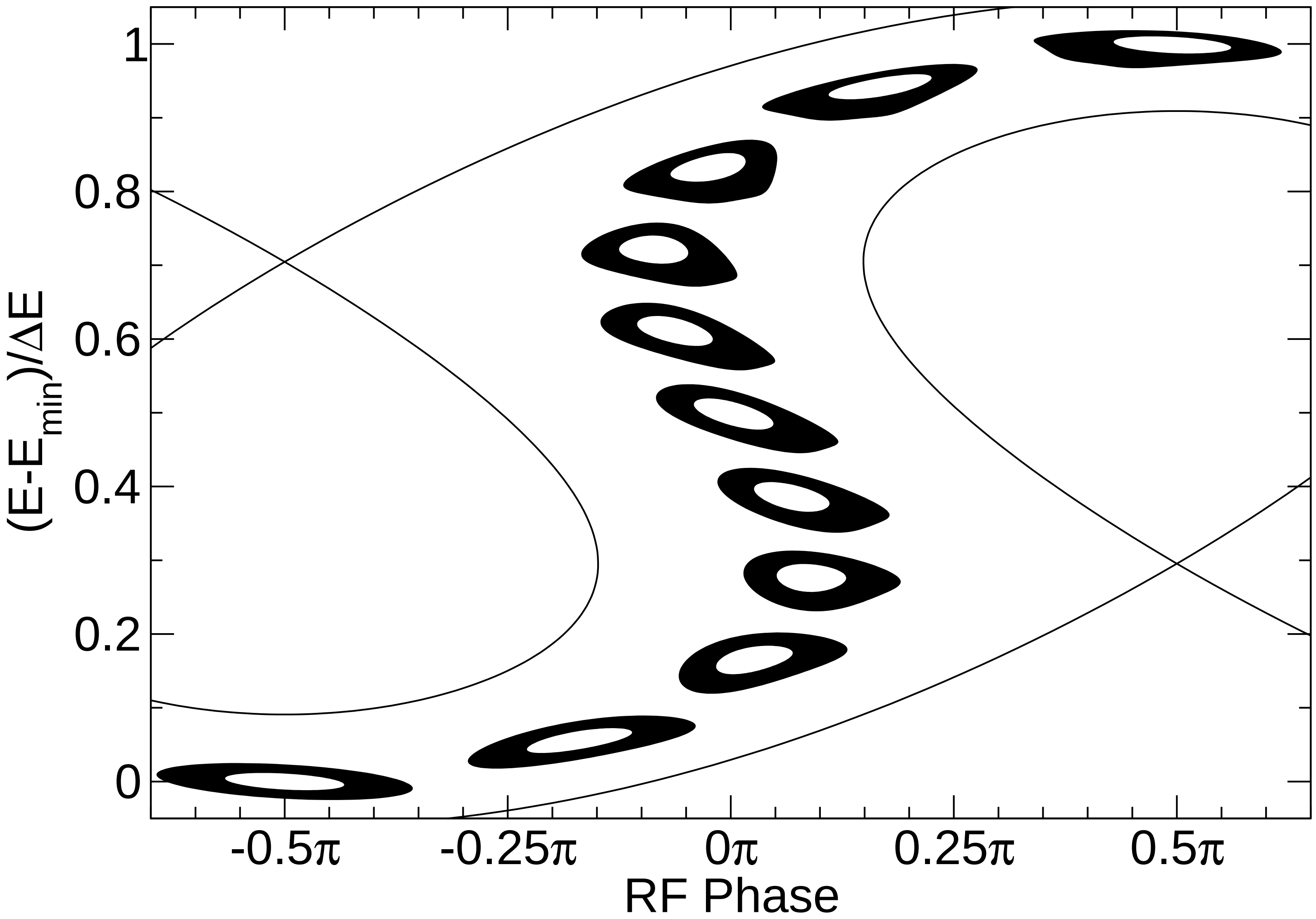
Optimal Orientation



After FFAG



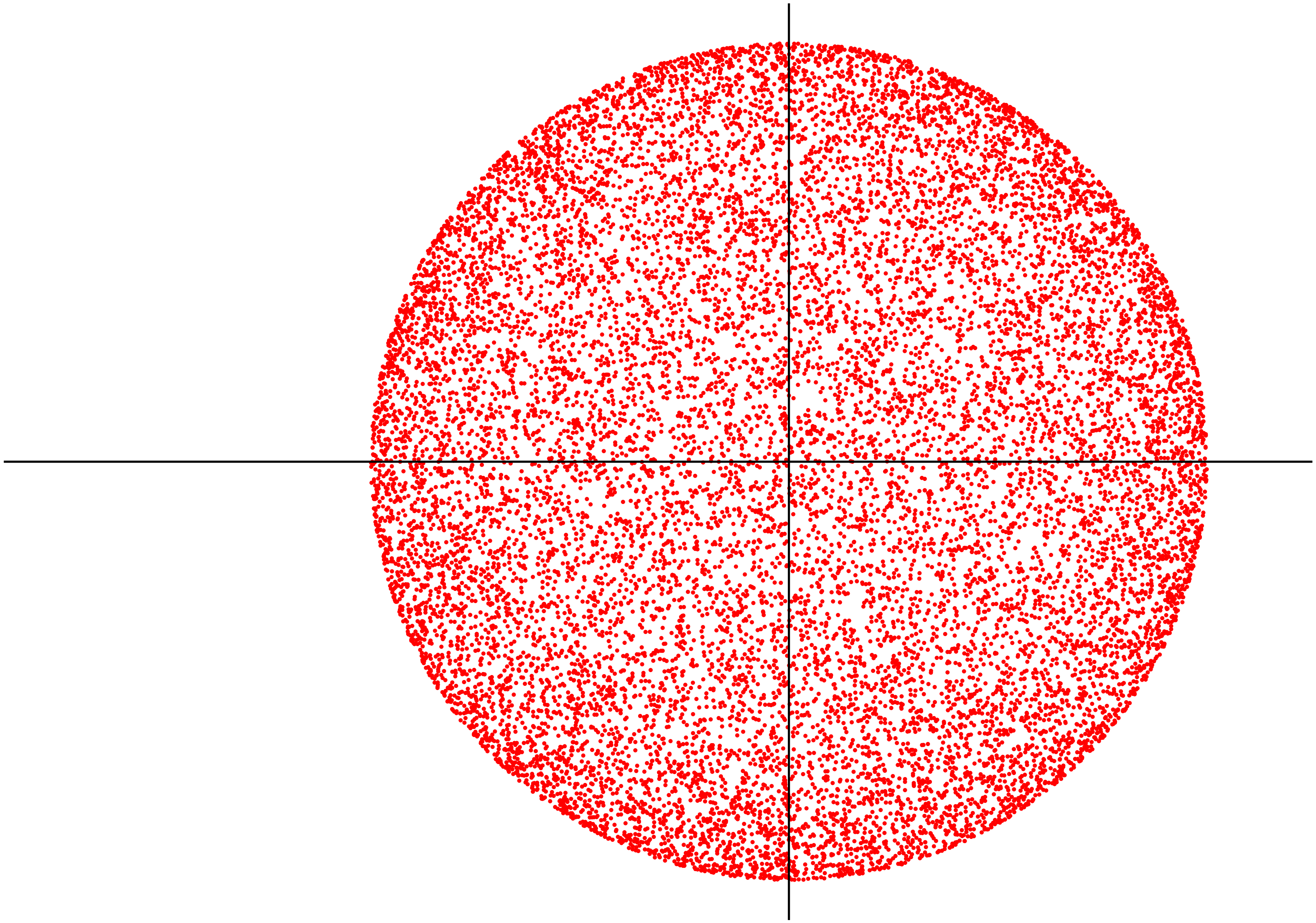
Bunch Being Accelerated



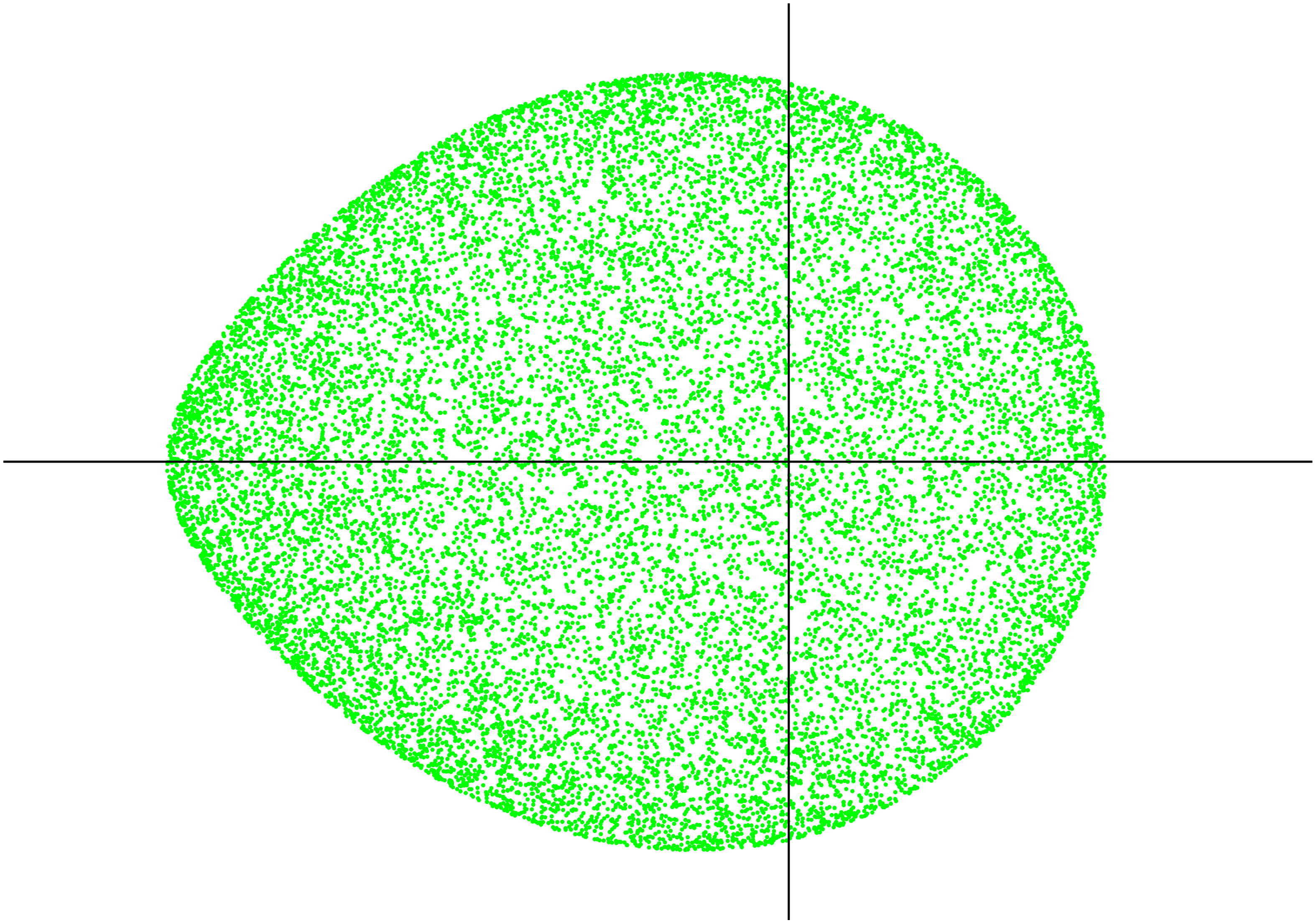
Emittance Reduction Example

- Before, found that for some cases to lowest order, emittance went down!
- What does this mean?
- Properly choose g_3 to get “emittance reduction”
- Nearly uniform distribution, but weighted slightly to the outside. 0.6% emittance reduction
- Distribution more heavily weighted to the outside: 6.3% emittance reduction

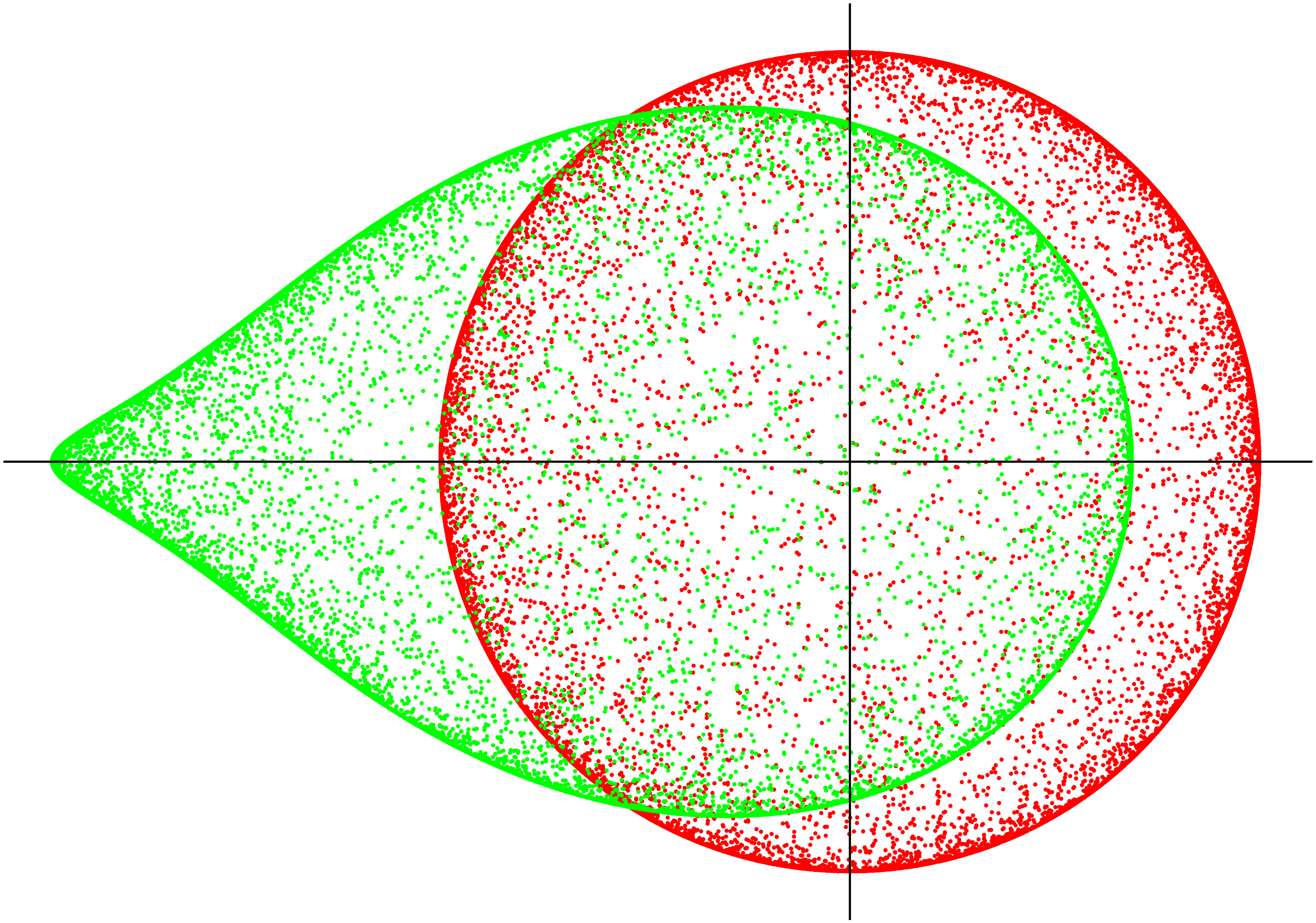
Nearly Uniform: Before



Nearly Uniform: After



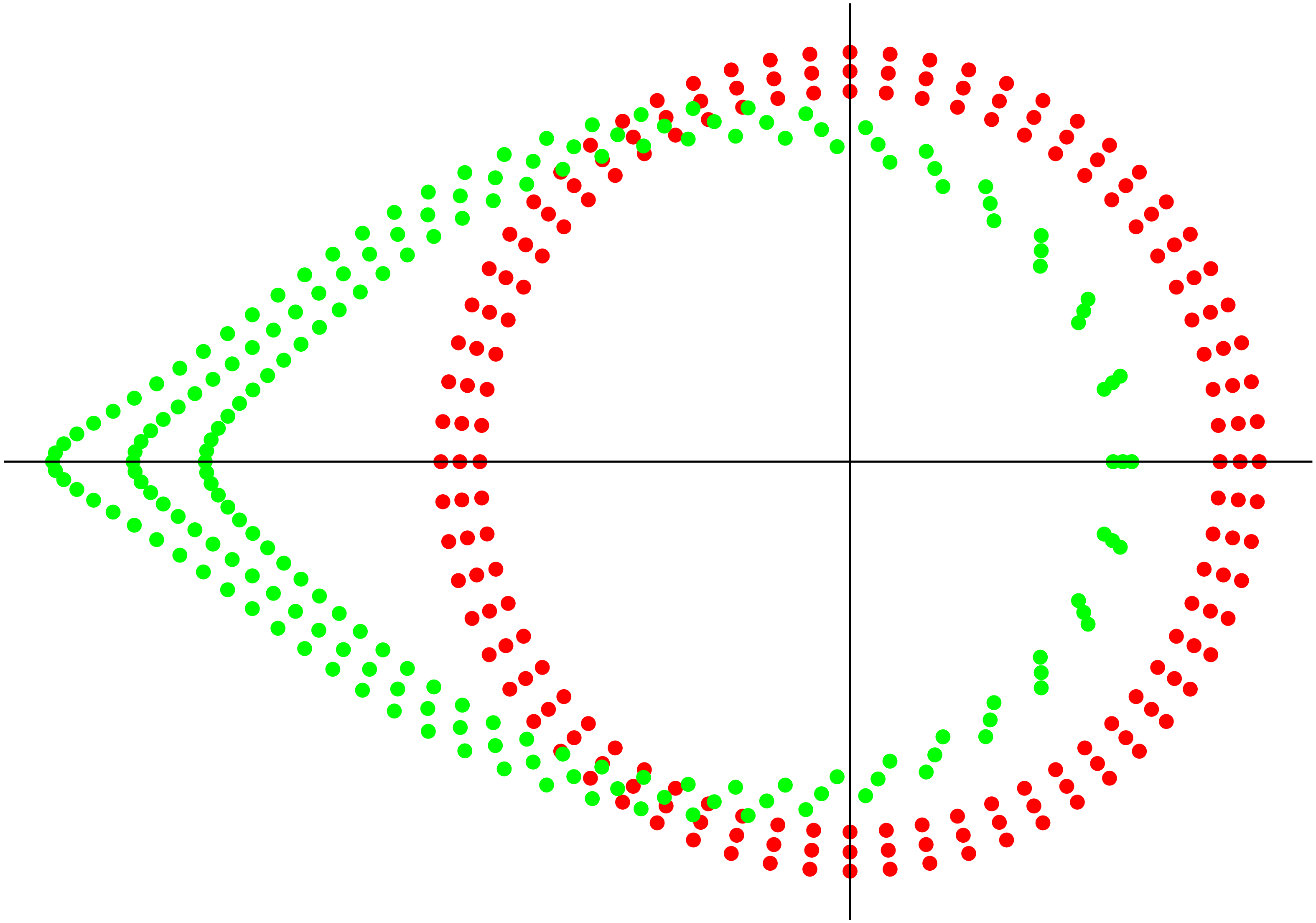
Ring Distribution



Emittance Reduction: Analysis

- Phase space area occupied and local density stay the same! No violation of phase space area conservation
  - Reduction is an artifact of the second-order covariance matrix computation
- Distribution is getting nonlinearly shifted toward the left center.
  - Particles are getting concentrated near that point, reducing computed emittance
  - With a more uniform distribution, particles are also pushed away from that point
  - Ring-like distribution has fewer particles being pushed away
- Emittance (from second-order covariance matrix) may not always characterize the desired behavior

Individual Particles



Ellipse Distortion

- Emittance growth seems to be a problematic criterion
- Potentially better criterion for FFAG performance: ellipse distortion
- Choose $\boldsymbol{w}_0$ and symmetric B with determinant 1 which minimizes

$$\sup \left|(\boldsymbol{w}-\boldsymbol{w}_0)^T B(\boldsymbol{w}-\boldsymbol{w}_0) - 2J\right|$$

  - $\boldsymbol{w}$ is phase space point at the end which started on an ellipse at the beginning
  - J is the “action” value on the original ellipse
- Result is first order in g_3

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Ellipse Distortion (cont.)

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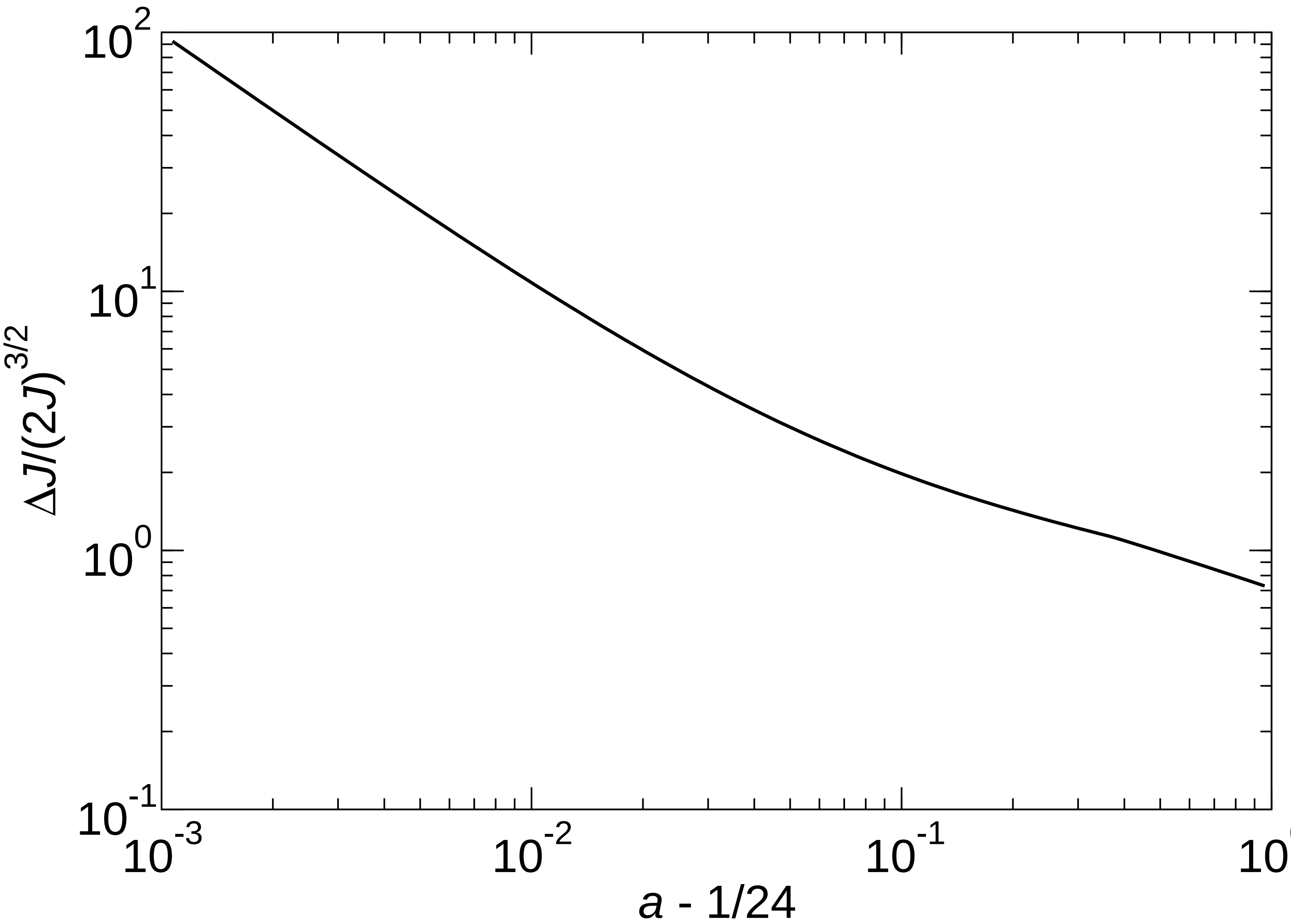
- As before, plot ellipse distortion vs. a
  - Minimized over initial ellipse orientation
  - Minimized over b
- Note different qualitative behaviors
  - Emittance growth was proportional to ϵ^2; action distortion is proportional to $(2J)^{3/2}$. Equivalently, radius distortion is proportional to r^2.
  - Coefficient is proportional to $(a - 1/24)^{-1}$, whereas for emittance growth it was $(a - 1/24)^{-2}$
  - Reasonable from above analysis

Ellipse Distortion vs. a



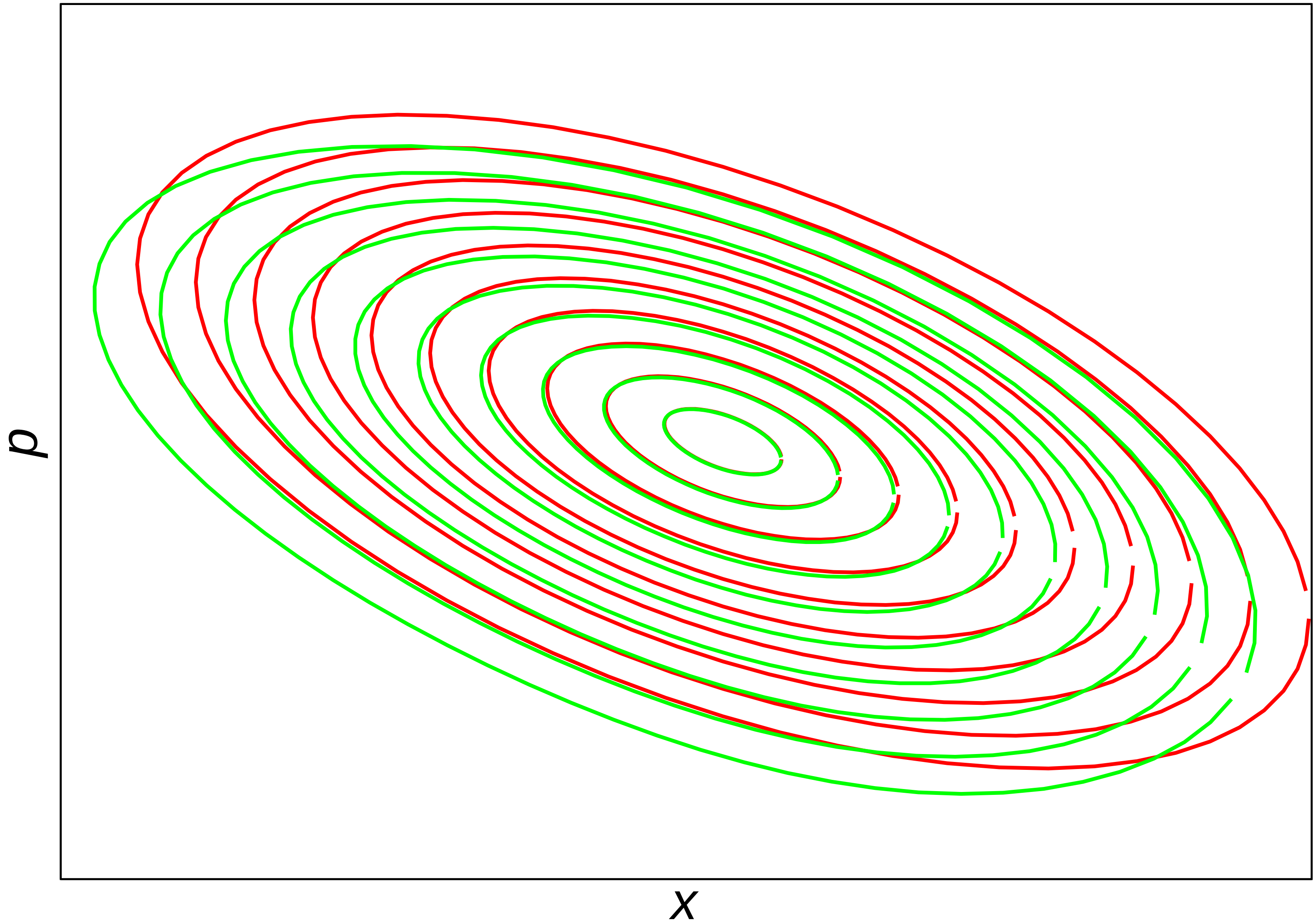
Ellipse Distortion with Shift

- Leaving out two effects
  - Amplitude-dependent shift of the ellipse center
  - Amplitude-dependent distortion of the ellipse shape
  - If we include these, then we don't care where the center of the ellipse is; we only care about the outer boundary enclosing all particles
- Now, trying to minimize

$$\sup \left| [\boldsymbol{w} - \boldsymbol{w}_0(J)]^T B(J) [\boldsymbol{w} - \boldsymbol{w}_0(J)] - 2J \right|$$

  - Center and matrix now depend on amplitude J
- Choose term first order in J in $\boldsymbol{w}_0(J)$ (2 parameters) and initial ellipse orientation (2 parameters) to kill lowest-order effect (4 parameters)

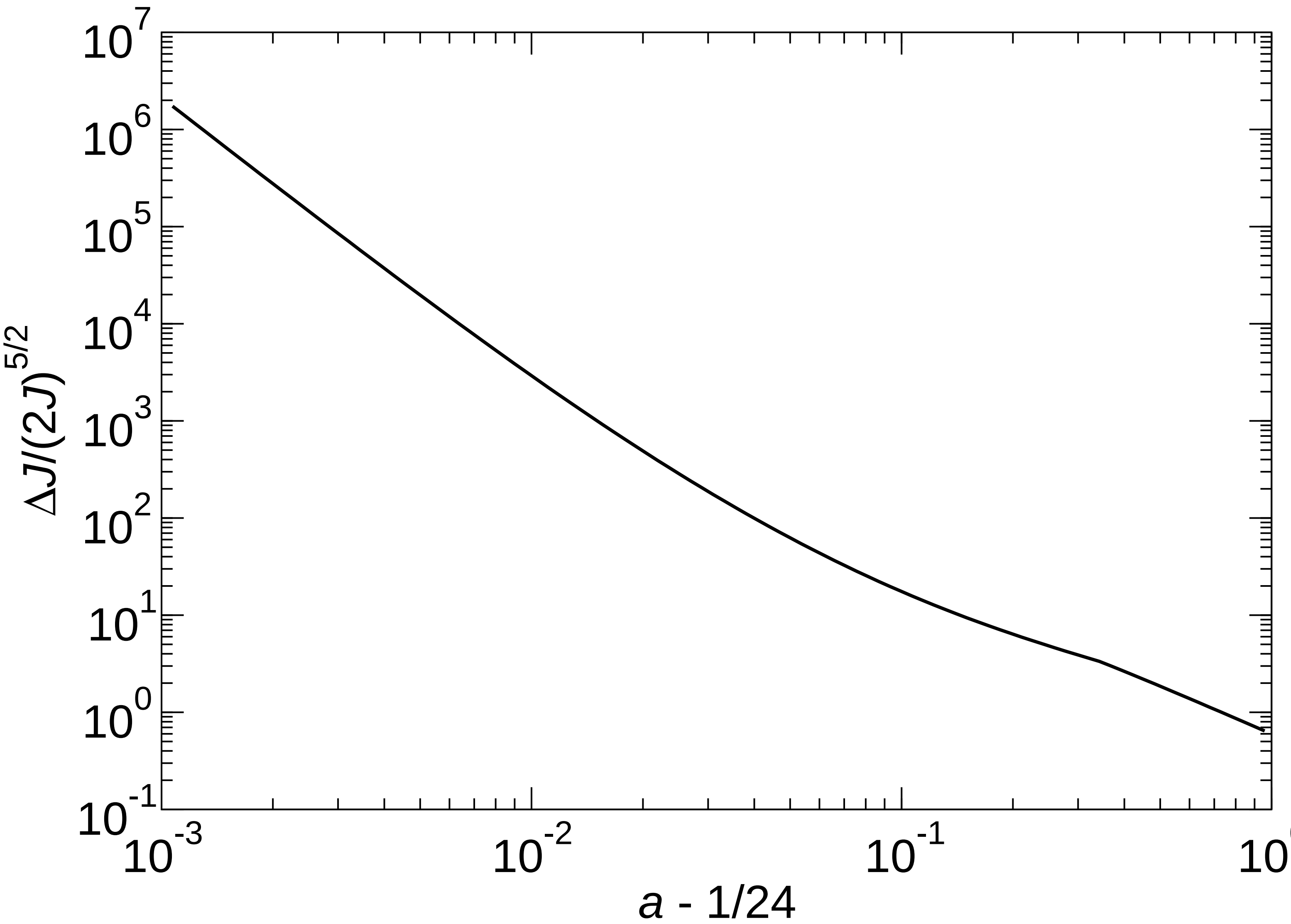
Ellipse Distortion vs. Amplitude



Ellipse Distortion with Shift

- The miracle: the term in $B(J)$ first order in J can eliminate the next order terms
  - Would not be true without our symmetry: we would have g_4
  - The symmetry breaking in the real machine is small: approximation may still be good
- Result is cubic in g_3 and first order in g_5
- Different characteristic behavior
  - $\Delta J \propto (2J)^{5/2}$, compared to $\Delta J \propto (2J)^{3/2}$ without shift removed, or $\delta\epsilon \propto \epsilon^2$ for emittance growth
  - $\Delta J \propto (a - 1/24)^{-3}$, compared to $\Delta J \propto (a - 1/24)^{-1}$ without shift removed, or $\Delta\epsilon \propto (a - 1/24)^{-2}$ for emittance growth
    - ⋆ Terms cubic in g_3 dominate when $a - 1/24$ small
- Good for neutrino factory: don't care about low amplitude particles
- May not be as good for collider

Ellipse Distortion, Shift Removed



Conclusions

- Have developed a method for characterizing longitudinal distortion in FFAGs
- Calculation can be used to determine machine design specifications from requirements on longitudinal ellipse distortion
- The technique is a general one, useful for characterizing the performance of single-pass systems (linacs, transfer lines)