Muon Collaboration

BEAM DYNAMICS OF IONIZATION COOLING CHANNELS

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TOPICS

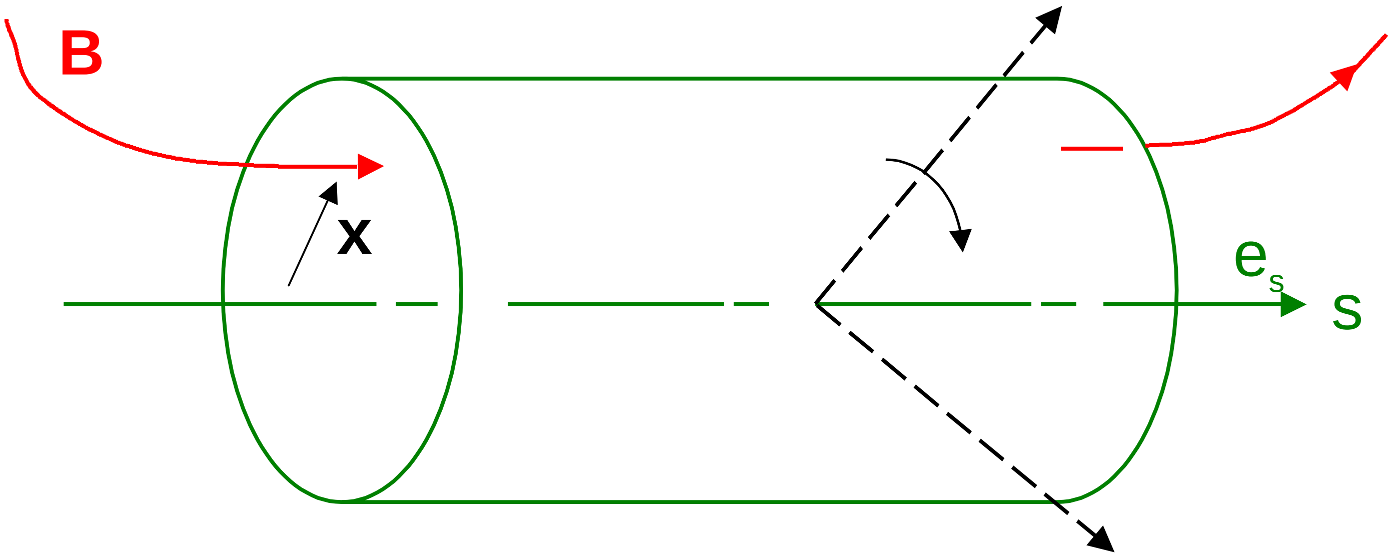
- Linear Transverse Cooling Dynamics

  Penn & Wurtele (PRL), Kim & Wang (PRL), Kim & Wang (MUCool)

- Orbit Stability and Energy Acceptance

  Wang & Kim (MUCool)

SOLENOIDAL FOCUSING



B(s): solenoidal field on s-axis

Simplest 3-D field satisfying $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{B} = 0$

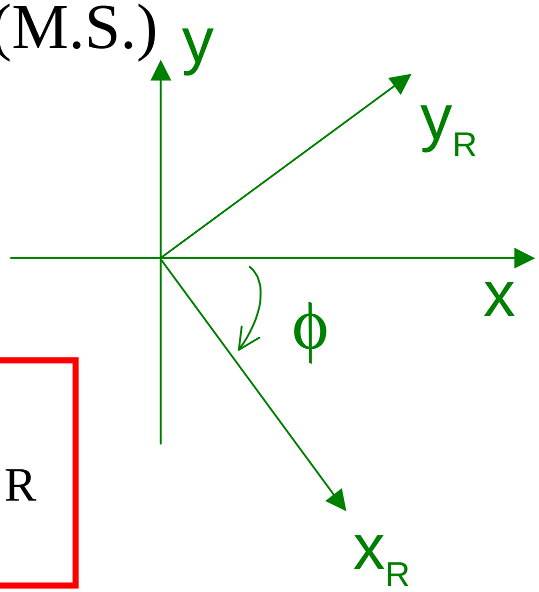
$$\boxed{\mathbf{B}(s, \mathbf{x}_\perp) = B(s)\mathbf{e}_s - \frac{1}{2}\left(\frac{dB}{ds}\right)\mathbf{x}_\perp + \text{non} - \text{linear terms}}$$

Equation of Motion

- Lab frame:

$$\frac{d}{ds}P_s\frac{d\mathbf{x}_\perp}{ds} = -qB\mathbf{e}_s\times\frac{d\mathbf{x}}{ds} - \frac{q}{2}B'\mathbf{e}_s\times\mathbf{x} - \eta P_s\frac{d\mathbf{x}}{ds} + P_s\mathbf{u}$$

$$\eta = \frac{1}{P_s}\frac{dP_s}{ds}\bigg|_{\mathrm{loss}}\quad ; \qquad \mathbf{u}:\ \text{stochastic excitation (M.S.)}$$



- Larmor frame:

$$\boxed{\frac{d}{ds}P_s\frac{d\mathbf{x}_R}{ds} = -P_s\kappa^2\mathbf{x}_R - P_s\eta\left(\frac{d\mathbf{x}_R}{ds} - \kappa\mathbf{e}_s\times\mathbf{x}_R\right) - P_s\mathbf{u}_R}$$

$$\frac{d\phi}{ds} = \kappa = \frac{qB}{2P_s}$$

ENVELOPE EQUATIONS IN LARMOR FRAME

Phase space variables: $\{\mathbf{x},\mathbf{x}'\} = \{x, y, x', y'\}$

$$\frac{d\mathbf{x}}{ds} = \mathbf{x}'$$

$$\frac{d\mathbf{x}'}{ds} = -\kappa^2\mathbf{x} - \eta(\mathbf{x}' - \kappa\mathbf{e}_s \times \mathbf{x}) - \mathbf{u}$$

$$\langle \mathbf{x}^2 \rangle = \varepsilon\beta \qquad \varepsilon = \sqrt{\langle \mathbf{x}^2 \rangle \langle \mathbf{x}'^2 \rangle - \langle \mathbf{x}\cdot\mathbf{x}' \rangle}$$

$$\langle \mathbf{x}'^2 \rangle = \varepsilon\gamma \qquad \beta\gamma - \alpha^2 = 1$$

$$\langle \mathbf{x}\cdot\mathbf{x}' \rangle = -\varepsilon\alpha$$

$$\mathbf{e}_s \cdot \langle \mathbf{x}\times\mathbf{x}' \rangle = \langle xy' - yx' \rangle = L$$

Excitation $\chi = \mathscr{n} \langle \mathbf{u}^2 \rangle = \left(\frac{13.6\ \mathrm{MeV}}{pv}\right)^2 \frac{1}{L_{Rad}}$

Evolution of "Lattice Functions"

$$\frac{d\beta}{ds}+2\alpha=\left(\eta\beta-\frac{\eta\kappa L+\chi}{\varepsilon}\beta^2\right)$$

$$\frac{d\alpha}{ds}+\frac{1+\alpha^2}{\beta}-\eta^2\beta=\left(-\frac{\eta\kappa L+\chi}{\varepsilon}\beta\alpha\right)$$

R.H.S. are small!

$$\Rightarrow\quad \frac{1}{2}\beta''+\kappa^2\beta-\frac{1}{\beta}\left(1+\frac{\beta'^2}{4}\right)=0$$

EVOLUTION OF "BEAM" PARAMETERS:

$$\frac{d}{ds}\underbrace{\begin{pmatrix}\varepsilon \\ L\end{pmatrix}}_{\psi} = -\begin{pmatrix}\eta & -\eta\kappa\beta \\ -\eta\kappa\beta & \eta\end{pmatrix}\begin{pmatrix}\varepsilon \\ L\end{pmatrix} + \underbrace{\begin{pmatrix}\beta\chi \\ 0\end{pmatrix}}_{\Xi}$$

General Solution:

$$\Psi(s) = e^{-\hat{\Gamma}(s)}\Psi(0) + e^{-\hat{\Gamma}(s)}\int_0^s d\bar{s}\, e^{-\hat{\Gamma}(s)}\Xi(\bar{s})$$

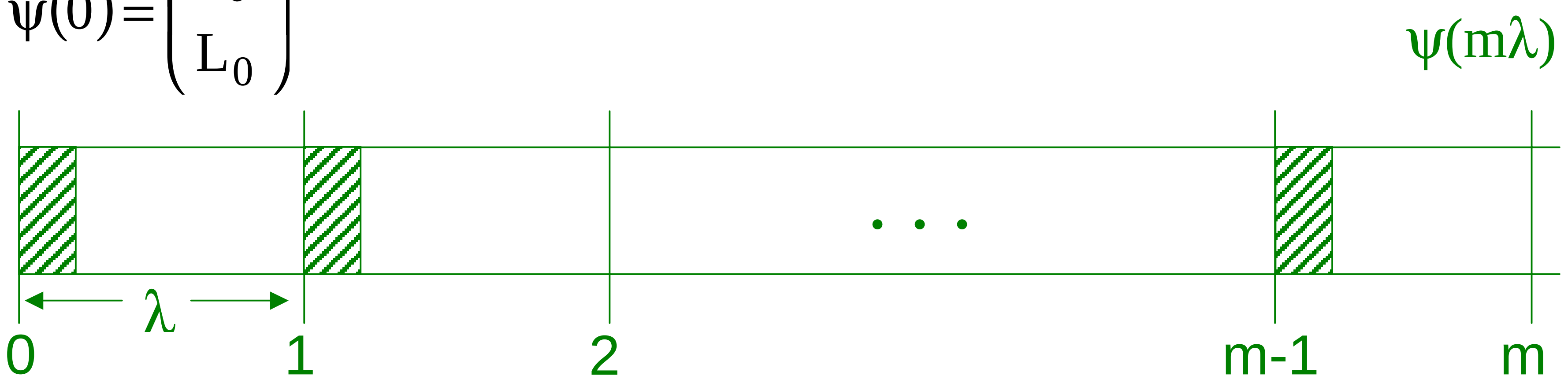
$$\hat{\Gamma}(s) \equiv \int_0^s d\bar{s}\hat{M}(\bar{s}) = \begin{pmatrix}\varsigma_1 & -\varsigma_2(s) \\ -\varsigma_2(s) & \varsigma_1(s)\end{pmatrix}$$

$$\varsigma_1(s) \equiv \int_0^s d\bar{s}\eta(\bar{s}), \quad \varsigma_2(s) \equiv \int_0^s d\bar{s}\eta(\bar{s})\kappa(\bar{s})\beta(\bar{s})$$

PERIODIC COOLING CHANNELS

$$\psi(m\lambda) = e^{-m\Gamma(\lambda)}\psi(0) + \left\{ e^{-\Gamma(\lambda)} + e^{-2\Gamma(\lambda)} + \cdots + e^{-(m-1)\Gamma(\lambda)} \right\} e^{-\Gamma(\lambda)} \int_0^{\lambda} d\bar{s}\, e^{\Gamma(s)} \begin{pmatrix} \beta\chi \\ 0 \end{pmatrix}$$

$$\psi(0) = \begin{pmatrix} \varepsilon_0 \\ L_0 \end{pmatrix}$$

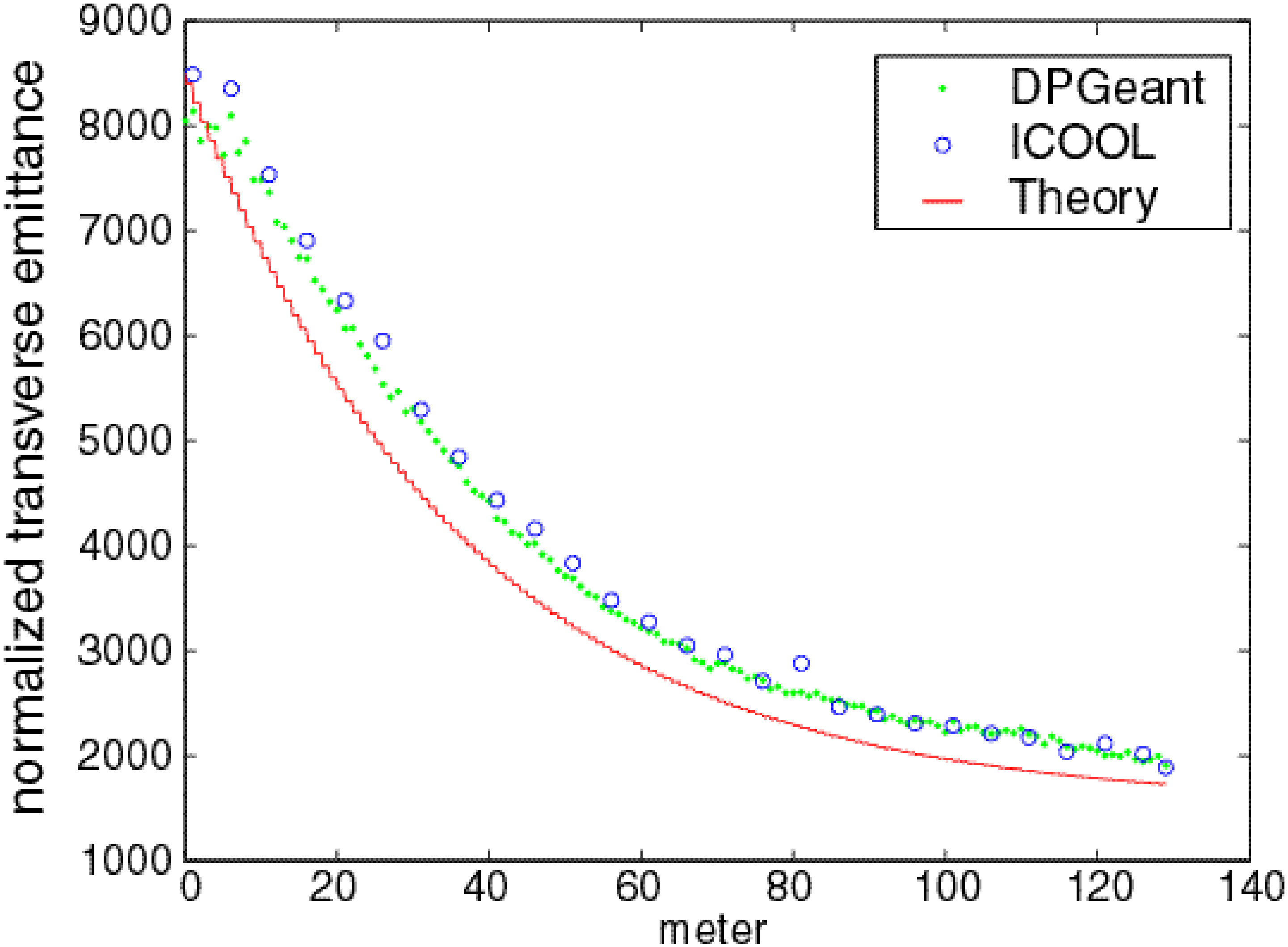


COMPARISON OF TRANSVERSE EMITTANCE (COOLING FACTOR) CALCULATIONS FOR MUON COOLING CHANNELS

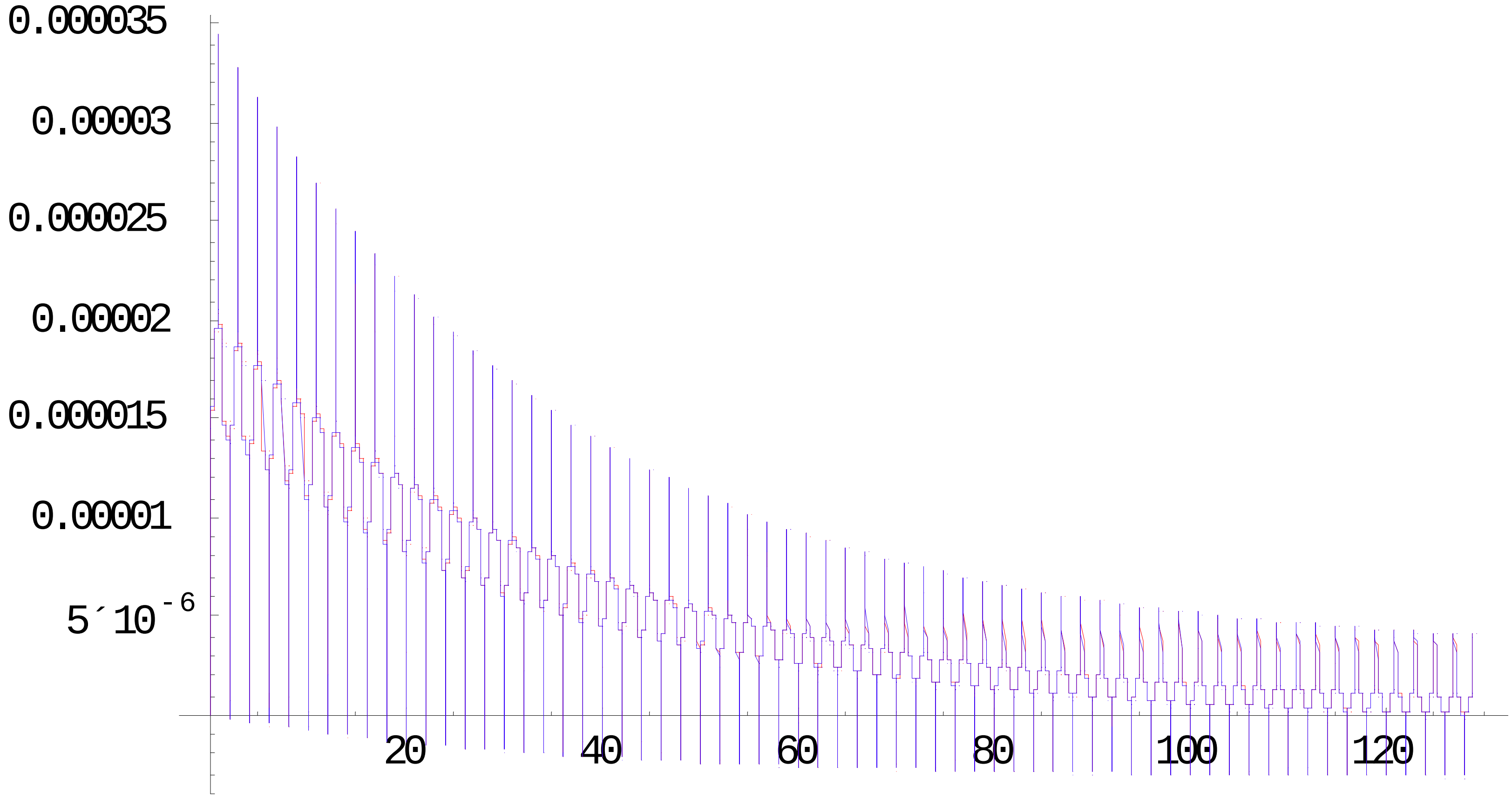
	3.4T FOFO (1)	3.4T FOFO (2)	DFOFO
ε^{∞}	0.19	0.27	0.080
$\varepsilon(m\lambda)$	0.21	0.34	0.095
ICOOL	0.22	0.34	0.10
DPGeant	0.24	0.37	N/A

$$\varepsilon^{\infty} \cong \frac{\int_0^{\lambda} d\bar{s}\,\beta(\bar{s})\chi(\bar{s})}{\int_0^{\lambda} d\bar{s}\,\eta(\bar{s})}$$

COMPARISON OF COOLING PERFORMANCE CALCULATION



CANONICAL ANGULAR MOMENTUM IN A FOFO CHANNEL



ANGULAR MOMENTUM MODIFIES NEUFFER EQUATION!

$$\boxed{\frac{d\varepsilon}{ds} = -\eta\varepsilon + \beta\chi} + \eta\kappa\beta L$$

Effect of angular momentum

Neuffer

$$\frac{dL}{ds} = -\eta L + \eta\kappa\beta\varepsilon$$

Equilibrium Emittance:

$$\varepsilon^{\infty} = \frac{\int_0^{\lambda} ds\beta(s)\chi(s)}{\int_0^{\lambda} ds\eta(s)} \mathscr{F}$$

$$\mathscr{F} = 1 \Big/ \left(1 - \left[\int_0^{\lambda} ds\eta\kappa\beta\right] \Big/ \left[\int_0^{\lambda} ds\eta\right]^2\right)$$

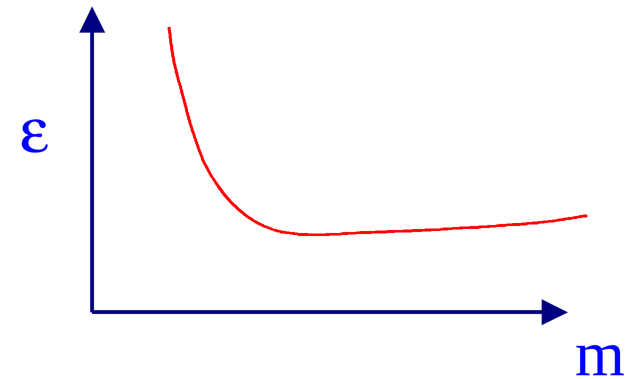
$\mathscr{F} = 1$ for alternating solenoids [R. Palmer]

MINIMUM EMITTANCE FOR SINGLE FLIP

Constant magnetic field

$$\kappa\beta = 1, \quad \mathscr{F} \to \infty$$

$$\varepsilon^{(m)} \to m \text{ (linearly divergent)}$$



Minimum emittance for $\varepsilon^{\infty} << \varepsilon(0)$

$$\varepsilon_{\min} \approx \varepsilon^{\infty}\left[\frac{3}{4} + \frac{1}{2}\ln\left(4\varepsilon(0)/\varepsilon^{\infty}\right)\right]$$

(ε^{∞} = equilibrium emittance for alternating case with the same β)

Single Flip Cooling Channel: Principle

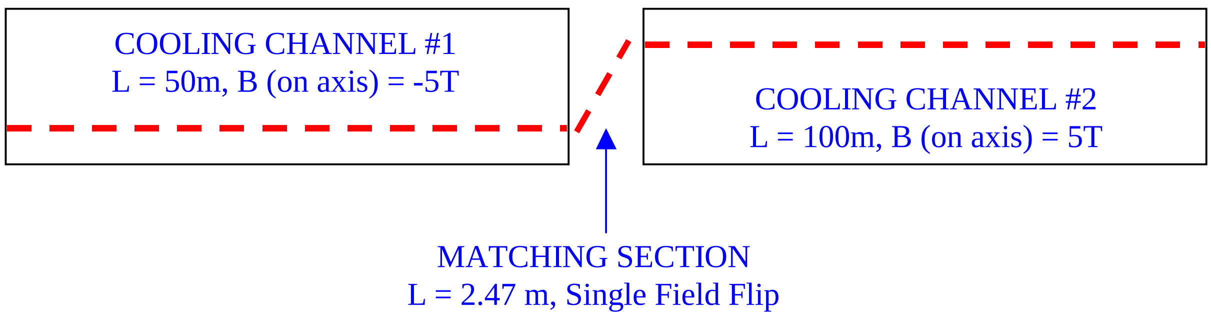


Figure 1: Schematic Layout of the Single Flip Channel

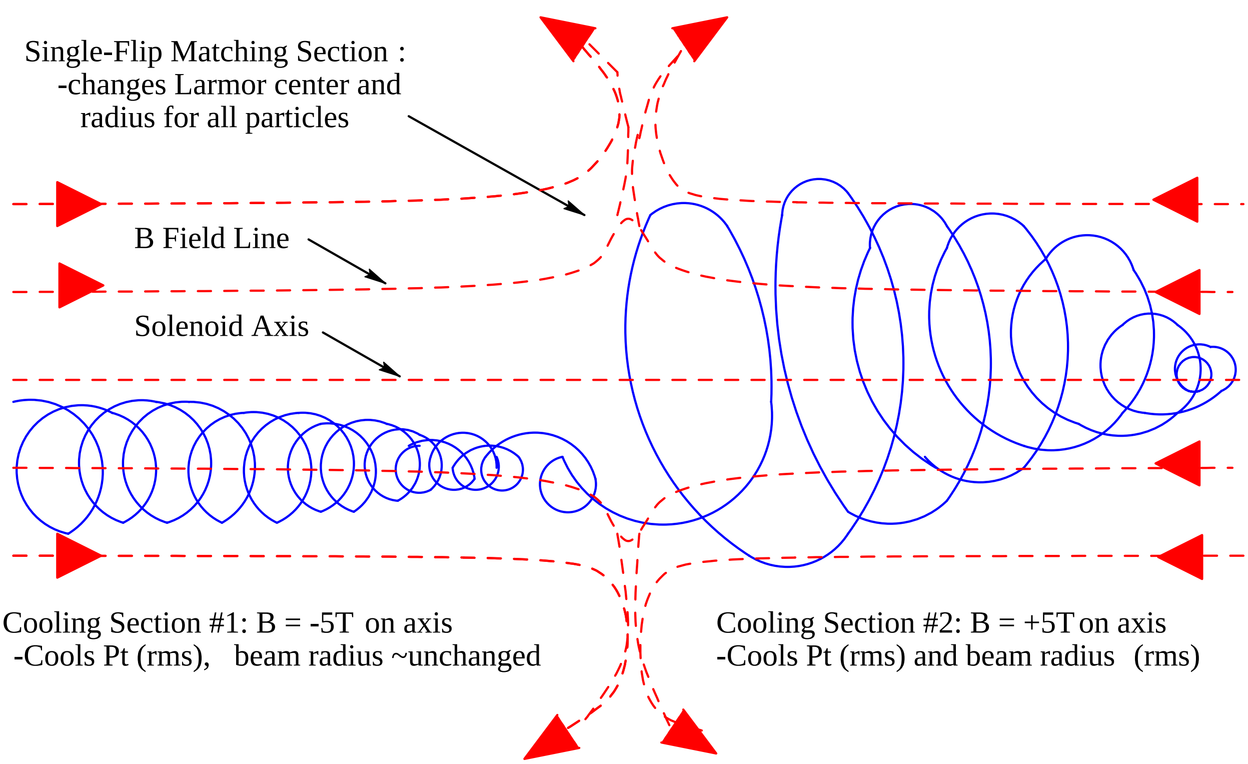
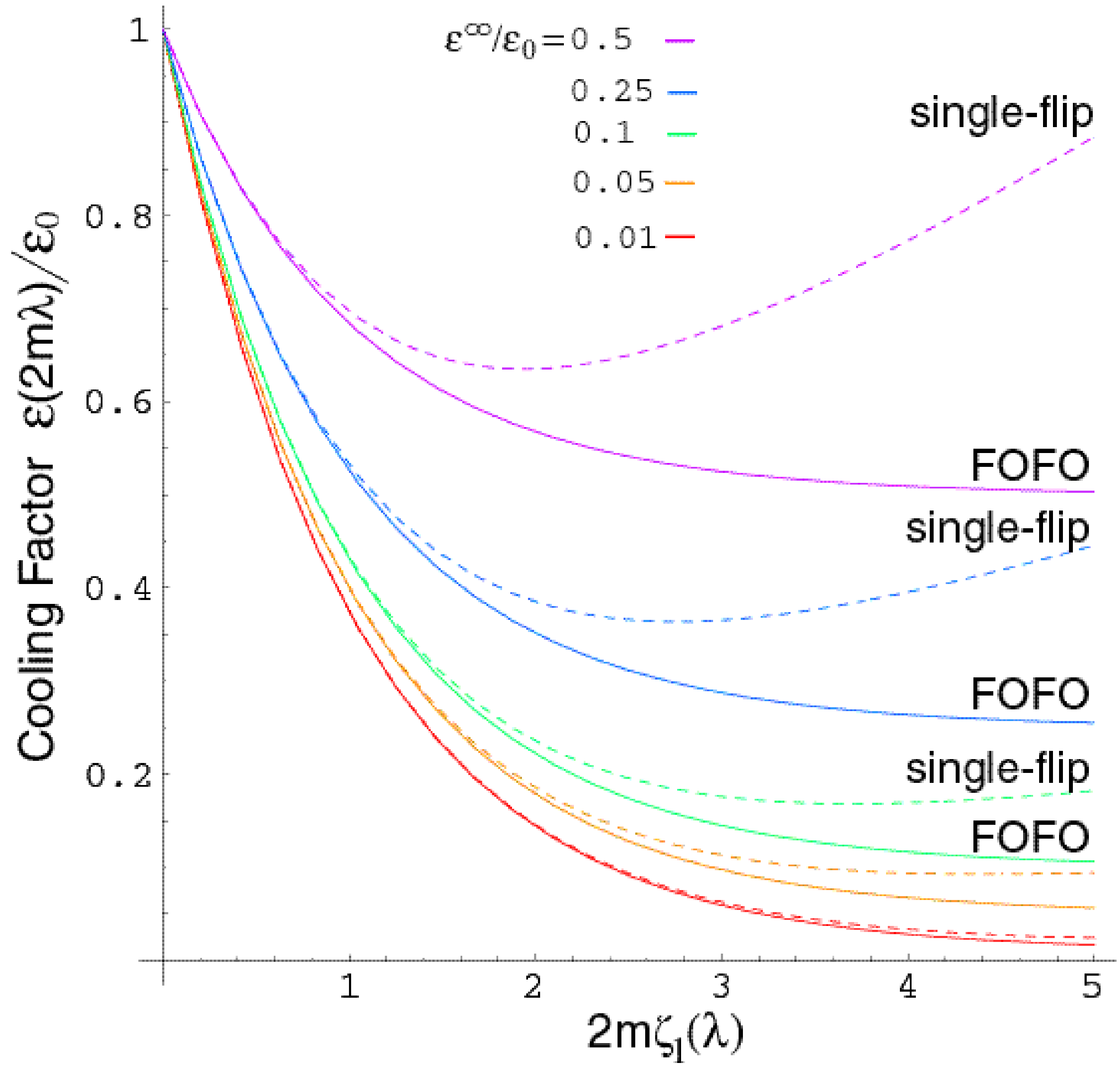


Figure 2: Particle Motion Cartoon in the Single Flip Channel

Jocelyn Monroe. FNAL

COMPARISON OF FOFO AND SINGLE-FLIP COOLING CHANNEL



HILL'S EQUATION

$$\frac{d^2\mathbf{x}}{ds^2} + \kappa^2(s)\mathbf{x}(s) = 0$$

$$\kappa^2(s) = \theta_0 + 2\theta_1 \cos 2s + 2\theta_2 \cos 4s + \ldots; \qquad \theta_0 = \left(\frac{qL}{2\pi P_s}\right)^2 \left\langle B_s{}^2 \right\rangle$$

G.W. Hill "On the Part of the Motion of the Lunar Perigee which is a Function of Mean Motion of Sun & Moon" (1886)

Whittaker & Watson

Magnus & Winkler

Stability Criteria

$$\operatorname{Tr} R(\pi) = \Delta = 2 - 4\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)\cdot D$$

$$D = \left\| \delta_{nm} + \frac{\theta_{n-m}}{\theta_0 - 4n^2} \right\| \qquad \text{(Hill's determinant)}$$

$$\Delta = 2\cos\left(\sqrt{\theta_0}\pi\right) + \frac{\pi \sin\sqrt{\theta_0}\pi}{2\sqrt{\theta_0}} \sum_{n=1}^{\infty} \frac{\left|\theta_n\right|^2}{\theta_0 - n^2} + \cdots$$

Stability $\left|\Delta\right| \le 2.$

Stopband centered at $\sqrt{\theta_0} \cong n\left(1 + \frac{5}{16}\frac{\theta_n{}^2}{\theta_0{}^2}\right)$

Width: $\Delta\sqrt{\theta_0} \cong n\frac{\theta_n}{\theta_0}$ $\left[\frac{\Delta P_z}{P_z} \approx \frac{\theta_n}{\theta_0}\right]$

FOFO

$B(s) = B_o \sin s$

$B^2(s) = B_o^{\ 2}\ (1 - \cos 2s)$

$\theta_0 = \theta_1 \ \rightarrow$ "maximally unstable"

To reduce θ_1

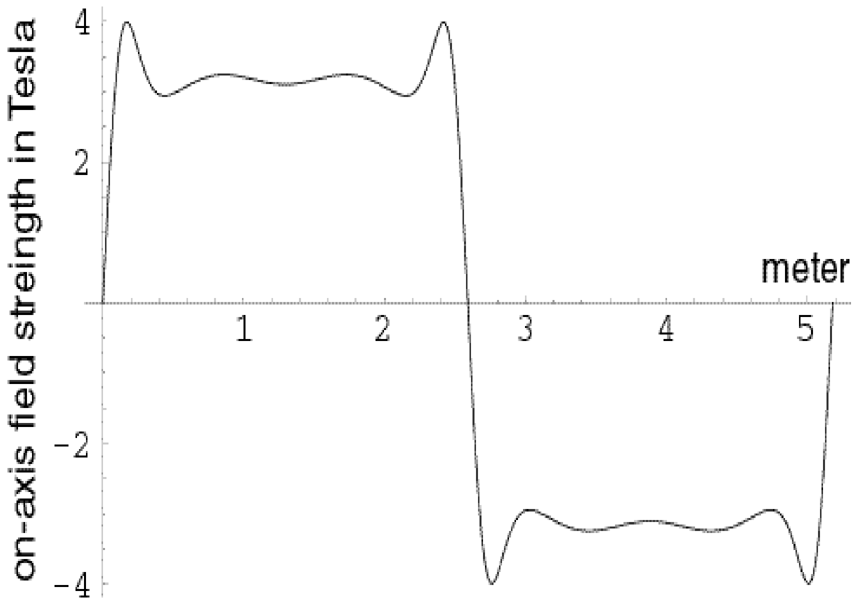
$B(s) = B_0\ (\sin s + a \sin 3s + \ldots)$

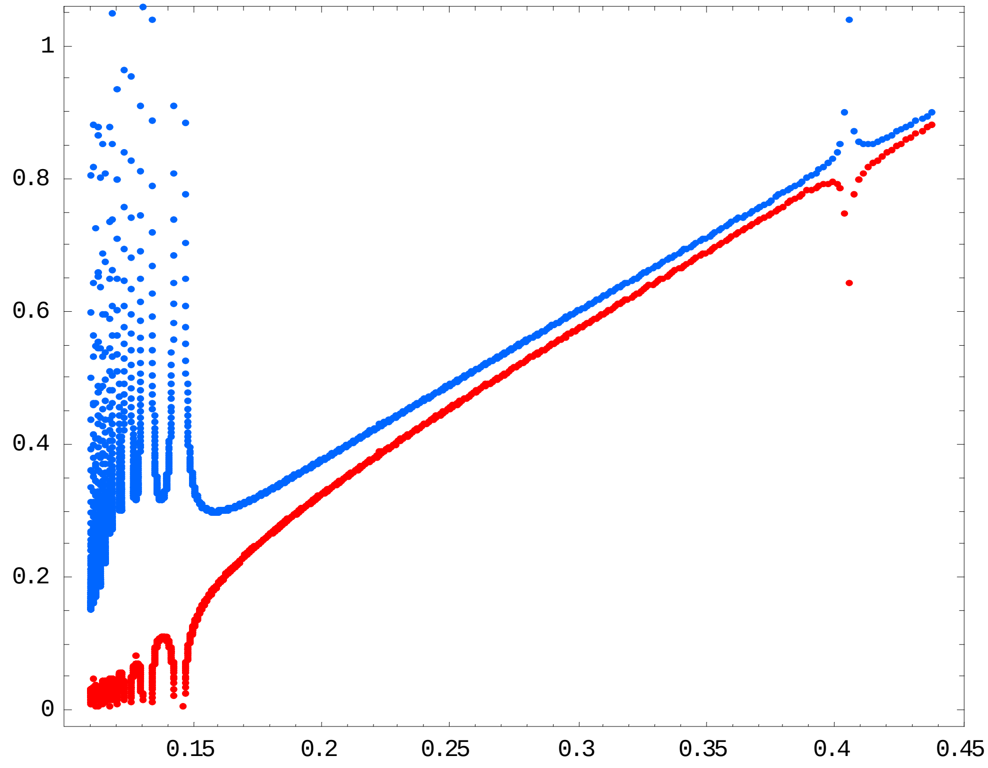
$B^2(s) = B_0^{\ 2}\ (\sin^2 s + a^2 \sin 3s + 2a \sin s \sin 3s + \ldots)$

$$B_0^{\ 2}\left(\frac{1-\cos 2s}{2} + \frac{a^2(1-\cos 6s)}{2} + a(\cos 2s - \cos 4s) + \ldots\right)$$

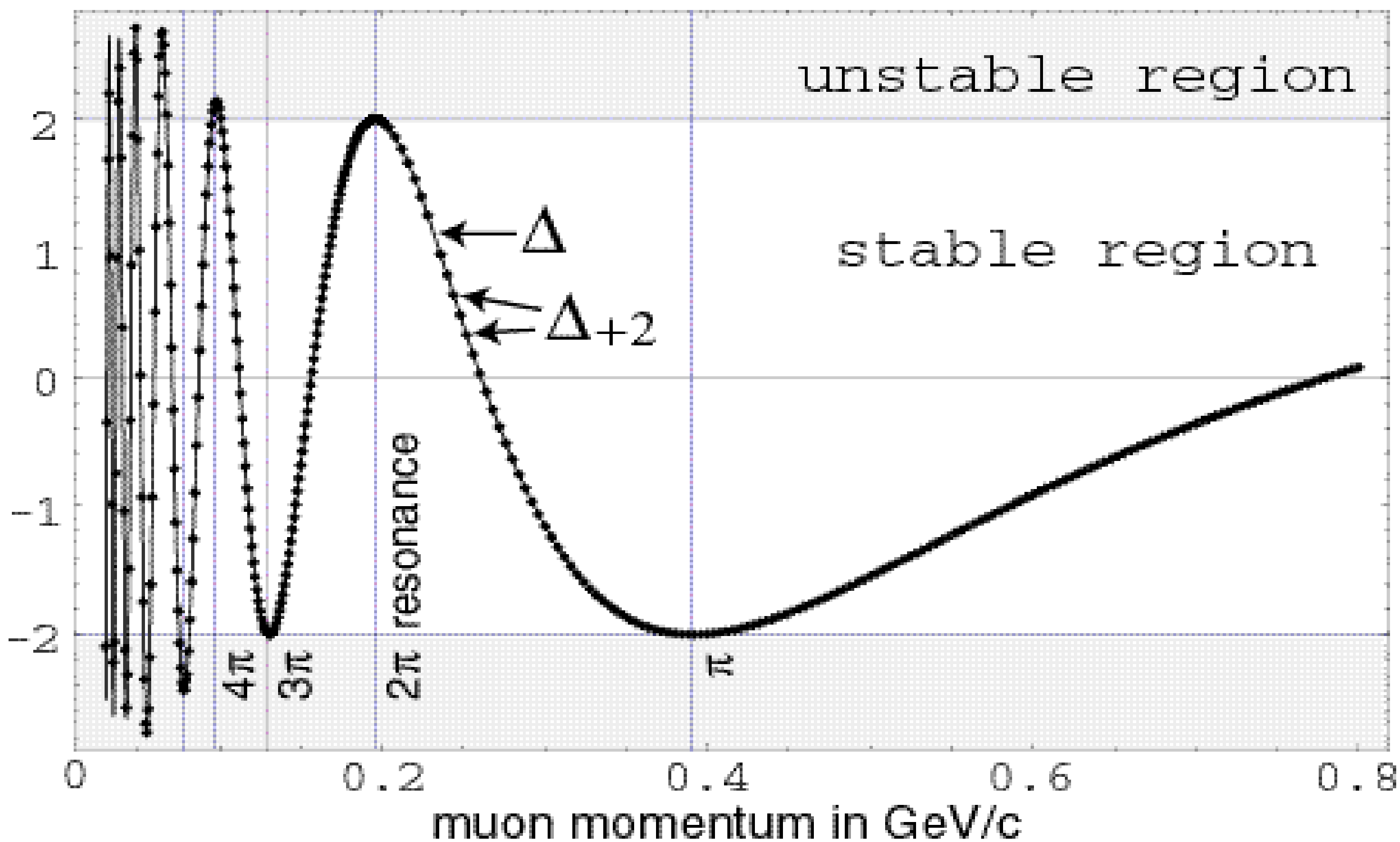
Choose $a = \dfrac{1}{2}$

V. BALBEKOV'S FAST-FLIP COOLING CHANNEL



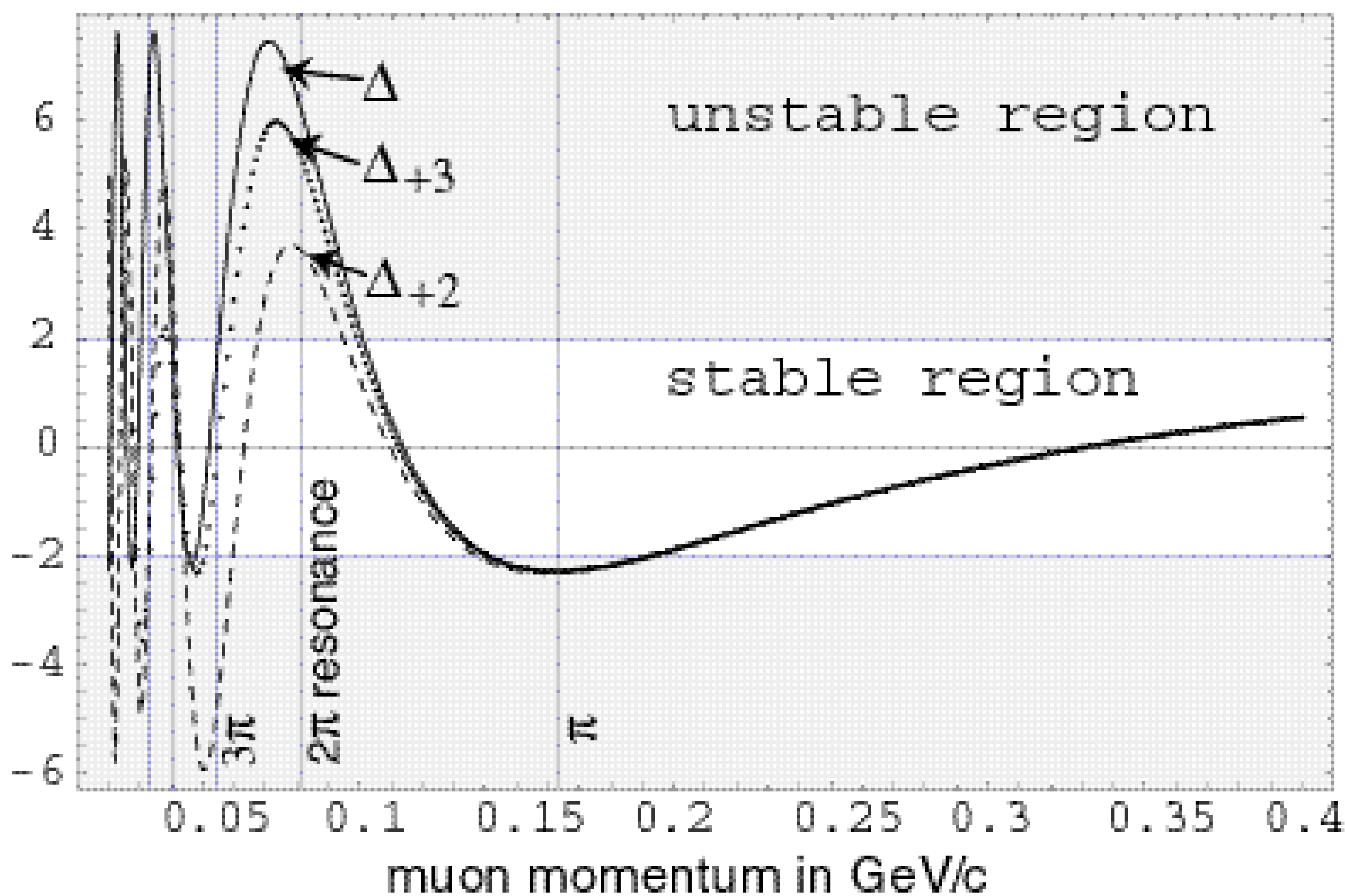


RESONANCE SUPPRESSION IN V. BALBEKOV'S CASE



At 200 MeV, $\{\kappa 0, \kappa 1, ..., \kappa 14\}$ =
{3.8186, -0.0039, -0.0047, -0.0060, -0.2239, -0.3203, -0.3345, -0.2976, -0.2383, -0.1759, -0.1212, -0.0783, -0.0477, -0.0272, -0.0146}

APPLICATION TO G. PENN'S SUPERFOFO EXAMPLE



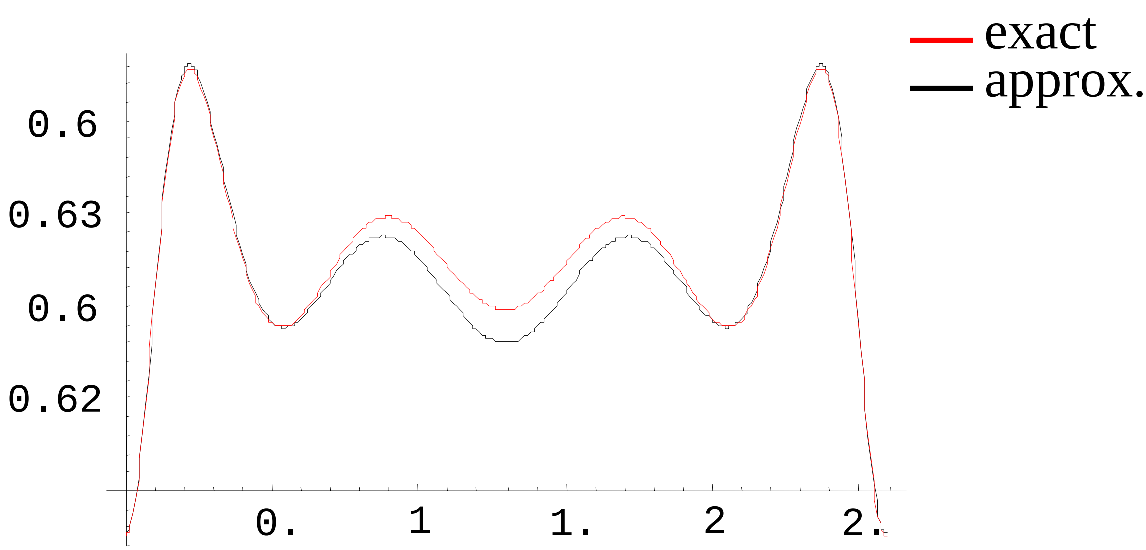
Learning to Optimize (minimize) Beta Function

$$\beta(s) = \frac{L \sin\left(\sqrt{\theta_0}\pi\right)}{\pi \sqrt{\theta_0}\,(\sin\psi)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\mathrm{Re}\left[\theta_n \ e^{i2ns}\right]}{n^2 - \theta_0} + \cdots \right\}$$

- Minimize L
- Large phase advance ψ
- Maximize $\theta_0 = \frac{Lq}{2\pi P_s}\sqrt{\left\langle {B_s}^2 \right\rangle}$ (higher pass band)
- Large θ_n to cancel 1

*All these within the stability limit!

V. Balbekov fast field flip channel at P=300MeV



SuperFOFO example of G. Penn at P=200 MeV