Subrelativistic Wakes Between Two Conducting Planes

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0.1 References

1. Fields of Particles and Beams Exiting a conductor, N.J.Carron; Am J. of Physics, submitted June 98).
2. Alex Chao Book
3. Alex Chao paper

0.2 Fields after a single plane

Carron gives the fields of a charge exiting perpendicularly from an ideal conducting plane. These fields are identical to those from the instantaneous creation of a pair of equal and opposite charges that then move at the same velocity, but in opposite directions. The anti-symmetry of the resulting fields assure that in the plane of symmetry there are no radial fields, and thus that the fields satisfy the boundry conditions of a perfect conducting sheet in that plane.

These fields, as given by Carron in cylindrical coordinates z, r, but written in MKS units and with the substitution of $\mathcal{R}_{+,-} = \gamma^3\ \mathrm{S}_{+,-}$ (the $\mathcal{R}$'s are the distances to the charges in their centers of mass system and thus gave more physical meaning than the S's) are given below, using the definitions:

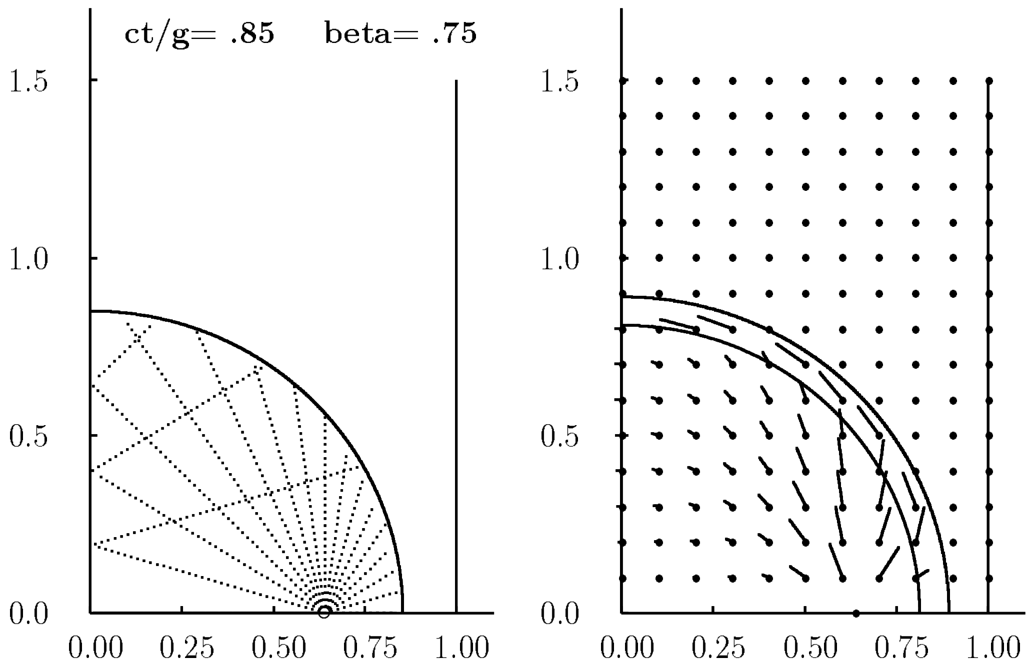
$$\begin{aligned} R &= \sqrt{r^2 + z^2} \\ \mathcal{R}_- &= \sqrt{\mathrm{r}^2 + (\gamma\,(\mathrm{vt} - \mathrm{z}))^2} \\ \mathcal{R}_+ &= \sqrt{\mathrm{r}^2 + (\gamma\,(\mathrm{vt} + \mathrm{z}))^2} \end{aligned}$$

$$\begin{aligned} \mathcal{E}_\mathrm{r} &= \frac{1}{4\pi\epsilon_\mathrm{o}} \left\{ \mathrm{q}\ \gamma^2 \left(\frac{\mathrm{r}}{\mathcal{R}_-^3} - \frac{\mathrm{r}}{\mathcal{R}_+^3} \right) \mathrm{u}(\mathrm{ct} - \mathrm{R}) \right. \\ &\quad \left. + \frac{2\beta\ q\ \sin\theta\ \cos\theta}{R\ (1 - \beta^2\ \cos^2\theta)}\ \delta(ct - R) \right\} \\ \mathcal{E}_\mathrm{z} &= \frac{1}{4\pi\epsilon_\mathrm{o}} \left\{ \mathrm{q}\ \gamma^2 \left(\frac{\mathrm{z} - \mathrm{vt}}{\mathcal{R}_-^3} - \frac{\mathrm{z} + \mathrm{vt}}{\mathcal{R}_+^3} \right) \mathrm{u}(\mathrm{ct} - \mathrm{R}) \right. \\ &\quad \left. + \frac{2\beta\ q\ \sin^2\theta}{R\ (1 - \beta^2\ \cos^2\theta)}\ \delta(ct - R) \right\} \\ B_\phi &= \frac{1}{4\pi\epsilon_o\ c} \left\{ q\ \gamma^2 \left(\frac{r}{\mathcal{R}_-^3} - \frac{r}{\mathcal{R}_+^3} \right) u(ct - R) \right. \\ &\quad \left. + \frac{2\beta\ q\ \sin\theta}{R\ (1 - \beta^2\ \cos^2\theta)}\ \delta(ct - R) \right\} \end{aligned}$$

In each equation, the first term is from the Coulomb fields of the moving charge, and the second term is the radiation emanating from the point where the charge were created (or exited the plane. Within the Coulomb term, the first subterm is from the charge its self, and the second from its image on the other side of the plane.

These are also the fields that are present with two planes, if the time $t < g/c$, where g is the distance between the planes; i.e. if it is early enough that the radiation has not yet reached the second plane:

The following first figure shows the radiation front, and electric field lines and from the individual charges: the one in the gap, and the image charge left of the plane. The total electric field is the sum of these two contributions, and is shown in the second figure. In this example $\beta_v = 0.75$, and the width of the radiation front has been artificially broadened to show the fields within it.



0.3 Fields with two planes

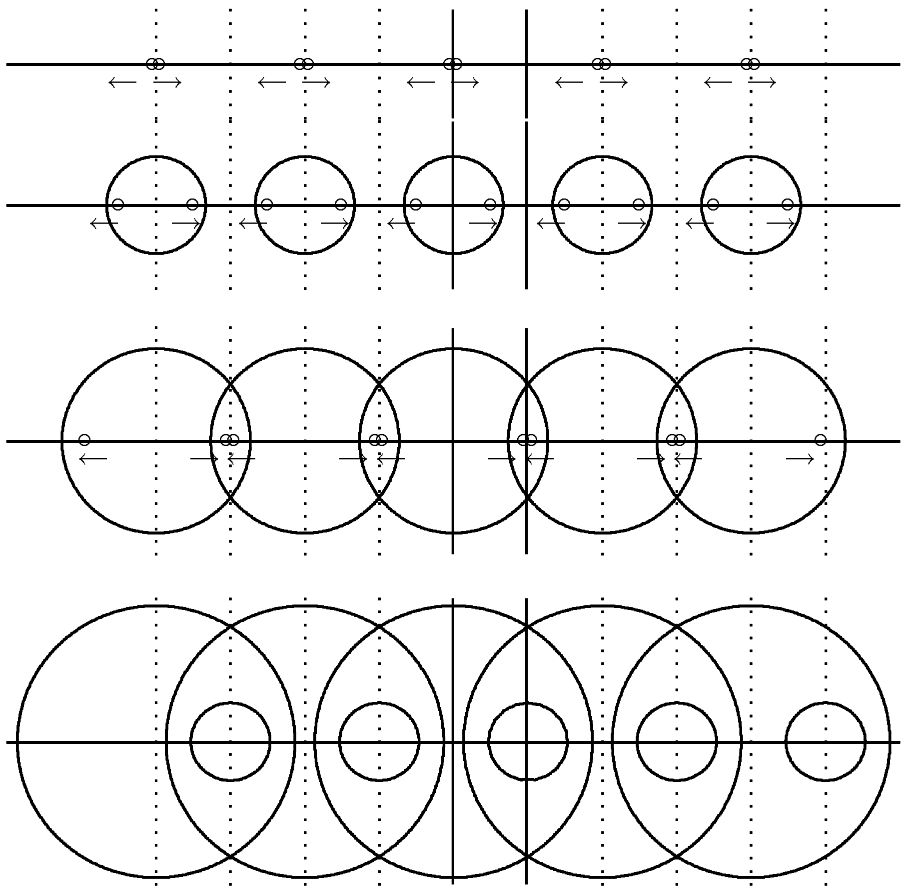
When a second plane is introduced after a gap g, there are several complications:

- There is a second radiation term from the disapearance of the charge as it enters the second plane at the time $t = g/(\beta c)$. It has the opposite sign to that from the creation, but is otherwise identical.

- There is another $u()$ in the Coulomb terms caused by the absence of the source charge after it leaves at time $t = g/(\beta c)$. Like the first $u()$, it moves out from the point of disapearance at a velocity c.

- To simultaneously satisfy the boundry conditions at the two planes, we must introduce an infinite series of images charges. These charges are created at an infinite number of points spaced by $2g$, travel in opposite directions, and simultaneoudly anihilate at an infinite number of points spaced by $2g$, but interleaved between the points of creation. The resulting series of charges have exact antisymmetry about the creation and anihilation points, and thus satisfy the boundry conditions for perfect conducting planes at all such planes.

  , generated by the original fields being reflected back and forth between the planes.



t=0.07
just after
emergence

t=0.67
half way

t=1.27
just before
exit

t=1.87
after
leaving

It will be convenient to give the fields in terms of dimensionless terms $\in_z$, $\in_r, \mathcal{B}$:

$$\begin{aligned}
\mathcal{E}_{\mathrm{z}} &= \frac{\mathrm{q}}{4\pi\epsilon_{\mathrm{o}}\ \mathrm{g}^2}\ \in_{\mathrm{z}} \\
\mathcal{E}_{\mathrm{r}} &= \frac{\mathrm{q}}{4\pi\epsilon_{\mathrm{o}}\ \mathrm{g}^2}\ \in_{\mathrm{r}} \\
B_\phi &= \frac{q}{4\pi\epsilon_o\ g^2\ c}\,\mathcal{B}_\phi
\end{aligned} \tag{1}$$

And redefine the radii as dimensionless numbers:

$$\begin{aligned}
R_o &= \frac{r}{g} \\
R_i &= \frac{\sqrt{r^2\ +\ (z+2ig)^2}}{g} \\
R_{i-} &= \frac{\sqrt{r^2\ +\ (z-g+2ig)^2}}{g} \\
\mathcal{R}_{\mathrm{i-}} &= \frac{\sqrt{\mathrm{r}^2\ +\ (\gamma\ (\mathrm{vt}-\mathrm{z}+2\mathrm{ig}))^2}}{\mathrm{g}} \\
\mathcal{R}_{\mathrm{i+}} &= \frac{\sqrt{\mathrm{r}^2\ +\ (\gamma\ (\mathrm{vt}+\mathrm{z}+2\mathrm{ig}))^2}}{\mathrm{g}}
\end{aligned} \tag{2}$$

And a normalaized time T and z dimension:

$$cT = \frac{ct}{g} \tag{3}$$

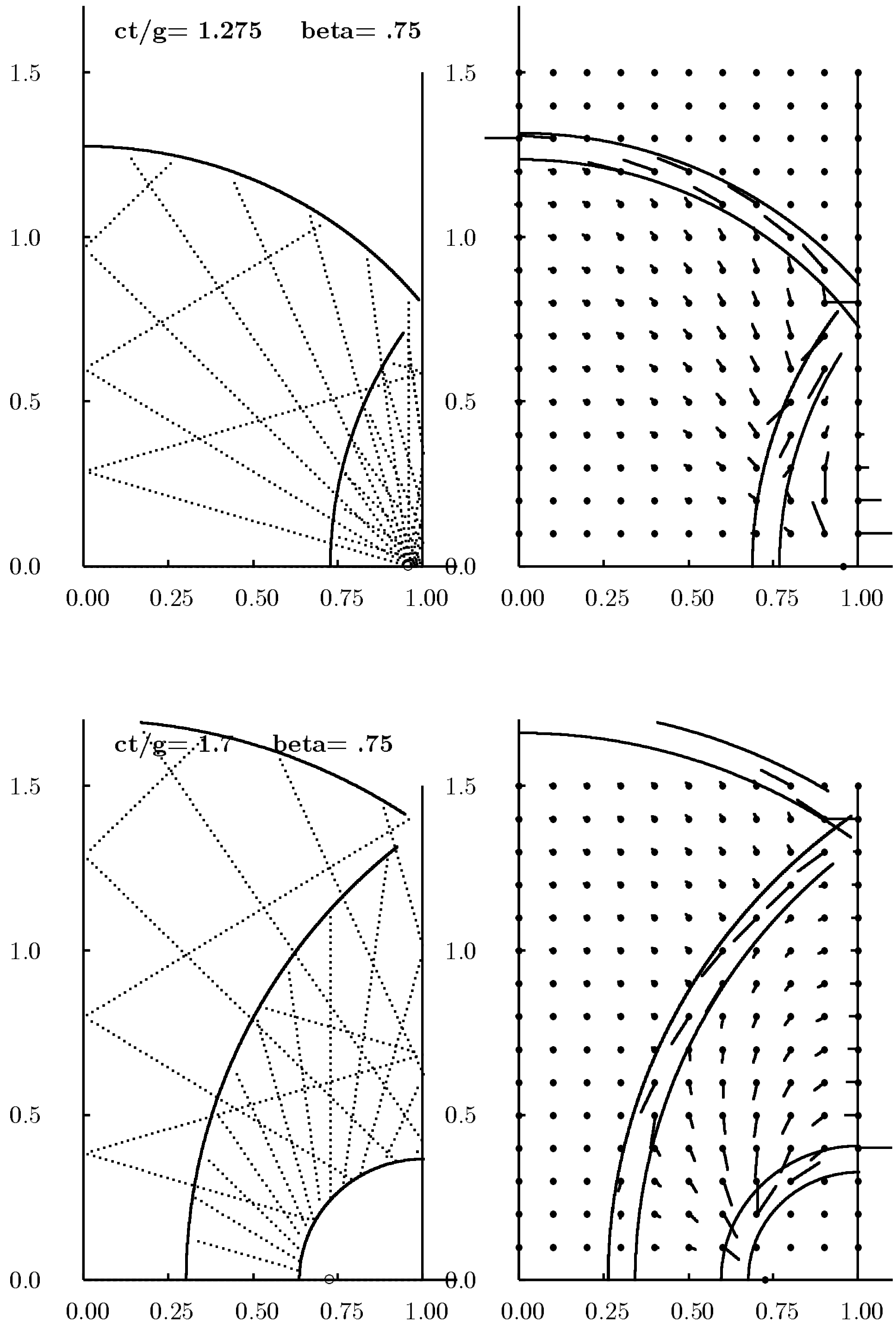
$$Z = \frac{z}{g} \tag{4}$$

The dimensionless terms $\in_z, \in_r, \mathcal{B}$ in the expressions for fields between the planes are:

$$\in_r = \sum_{i=-\infty}^{+\infty} \left\{ \gamma^2 \left(\frac{R_o}{\mathcal{R}^3_{\mathrm{i-}}} - \frac{R_o}{\mathcal{R}^3_{\mathrm{i+}}} \right) u(cT - R_i)\, u(R_{i-} - cT + \frac{1}{\beta}) \right.$$
$$+ \frac{2\beta \ \sin\theta \ \cos\theta}{R_i \ (1 \ - \ \beta^2 \ \cos^2\theta)} \ \delta(cT - R_i)$$
$$\left. - \frac{2\beta \ \sin\theta \ \cos\theta}{R_{i-} \ (1 \ - \ \beta^2 \ \cos^2\theta)} \ \delta(cT - \frac{1}{\beta} - R_{i-}) \right\} \tag{5}$$

$$\in_z = \sum_{i=-\infty}^{+\infty} \left\{ \gamma^2 \left(\frac{Z + 2i - \beta T}{\mathcal{R}^3_{\mathrm{i-}}} - \frac{Z + 2i + \beta T}{\mathcal{R}^3_{\mathrm{i+}}} \right) u(cT - R_i)\, u(R_{i-} - cT + \frac{1}{\beta}) \right.$$
$$+ \frac{2\beta \ \sin^2\theta}{R_i \ (1 \ - \ \beta^2 \ \cos^2\theta)} \ \delta(cT - R_i)$$
$$\left. - \frac{2\beta \ \sin^2\theta}{R_{i-} \ (1 \ - \ \beta^2 \ \cos^2\theta)} \ \delta(cT - \frac{1}{\beta} - R_{i-}) \right\} \tag{6}$$

$$\mathcal{B}_\phi = \sum_{i=-\infty}^{+\infty} \left\{ \gamma^2 \left(\frac{R_o}{\mathcal{R}^3_{\mathrm{i-}}} - \frac{R_o}{\mathcal{R}^3_{\mathrm{i+}}} \right) u(cT - R_i)\, u(R_{i-} - cT + \frac{1}{\beta}) \right.$$
$$+ \frac{2\beta \ \sin\theta}{R_i \ (1 \ - \ \beta^2 \ \cos^2\theta)} \ \delta(cT - R_i)$$
$$\left. - \frac{2\beta \ \sin\theta}{R_{i-} \ (1 \ - \ \beta^2 \ \cos^2\theta)} \ \delta(cT - \frac{1}{\beta} - R_{i-}) \right\} \tag{7}$$

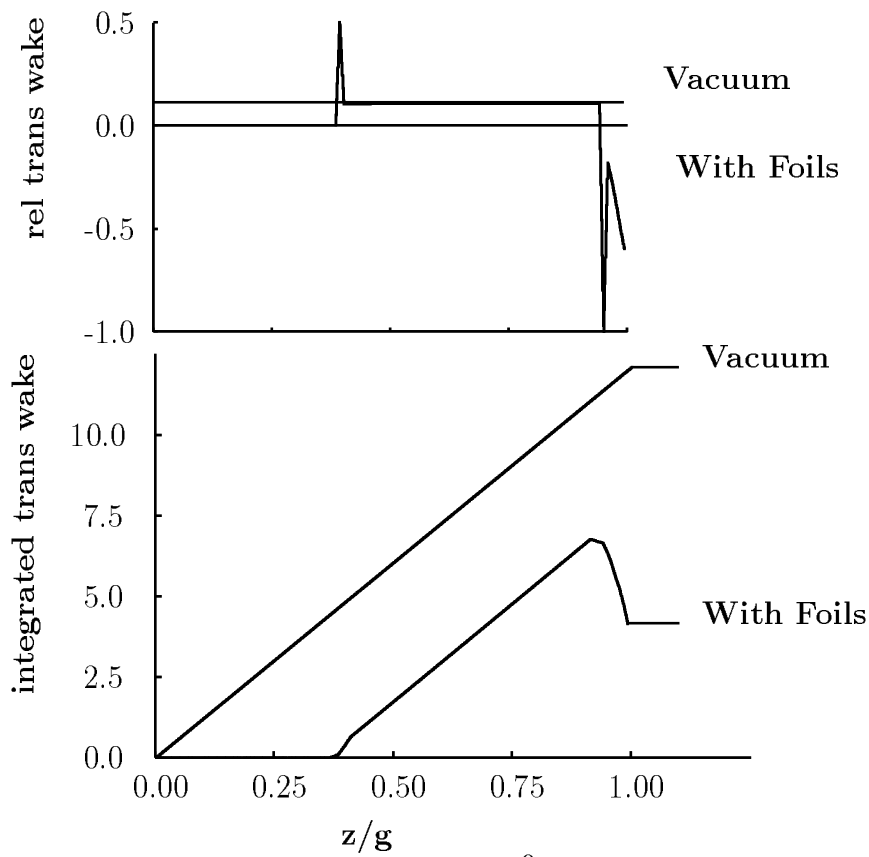

ct/g= 1.275 beta= .75
ct/g= 1.7 beta= .75
1.5
1.0
0.5
0.0
0.00
0.25
0.50
0.75
1.00

0.4 Wake Fields

The wake fields are the average of these fields, for a point at radius r and fixed z distance s behind the moving charge.

The next figure shows the relative values of the transverse forces as a function of time, as a particle enters and leaves the gap. The particle chosen is in the center of the bunch in the longitudinal direction and has a radial displacement equal to a typical beam sigma/g of 0.2. The red spikes are the contributions from the radiation terms. The second figure shows the integral of these fields, whose value, when divided by the gap, gives the transverse wake seen by this particle. The same quantities for the vacuum case, where there are no foils or walls, is also given.

It is seen that there is a delay before any fields arrive at the test charge. The particle then sees the radiation field, followed by coulomb fields similar to those in vacuum. They are relatively small because the electric and magnetic fields almost cancel. Near the end the particle encounters the reflected radiation followed by the reflected Coulomb fields. These are larger because the magnetic and electric effects now add, and they are of the opposite sign to the normal fields. It is the presence of these reflected fields, combined with the initial absence of any fileds, that reduces the final wake.



Again we define these wake fields in terms of dimensionless terms:

$$
\begin{aligned}
W_{\parallel} &= \frac{1}{4\pi\epsilon_o\ g^2}\ \mathcal{W}_{\parallel} \\
W_{\perp} &= \frac{1}{4\pi\epsilon_o\ g^2}\ \mathcal{W}_{\perp}
\end{aligned} \tag{8}
$$

And the dimensionless terms in the above are obtained by integration of the forces on a test charge ar a radius R_o and position a distance s behind the source charge:

$$
\mathcal{W}_{\parallel}(R_o,\ S) \ = \ \int_o^1 \in_z (R_o,\ Z - S)\ dZ \tag{9}
$$

$$
\mathcal{W}_{\perp}(R_o,\ S) \ = \ \int_o^1 \{\in_r (R_o,\ Z - S) \ - \ \mathcal{B}_{\phi}(R_o,\ Z - S)\beta\}\ dZ \tag{10}
$$

The integrations include integration over the δ functions, which depend on the relative absolute velocity beteen the integration variable dz and the advancing wave front. Numerically we can avoid this complication by repacing the δ functions by u functions over a sufficiently small interval ϵ:

$$
\delta(x) \ \rightarrow \ \frac{1}{\epsilon}\ u(x - \epsilon)\ u(-x + \epsilon)
$$

The wake fields are obtained by forming a grid of points in $r,\ z$ that moves with the charged particle. At each time step, the force on a test charge at that grid point is summed, and an average force obtained.

0.5 Gaussian Distributions

These functions are for point charges and have infinities at $R = 0$ and $S = 0$. Indeed the self retarding wake on the charge is itself infinite and the charge could not leave the ideal, infinitely conducting, uniform, conductor. These are not physical fields. Or course a single point charge can pass through a metal plane. The finite conductivity and non-uniformity of the conductor must be taken into account in this case.

However, for Gaussian distributions of charge in r and z the fields are nowhere infinite and can be taken as physical. The modifications due to finite conductivity and non-unifiormity of the conductors are now small and can be neglected.

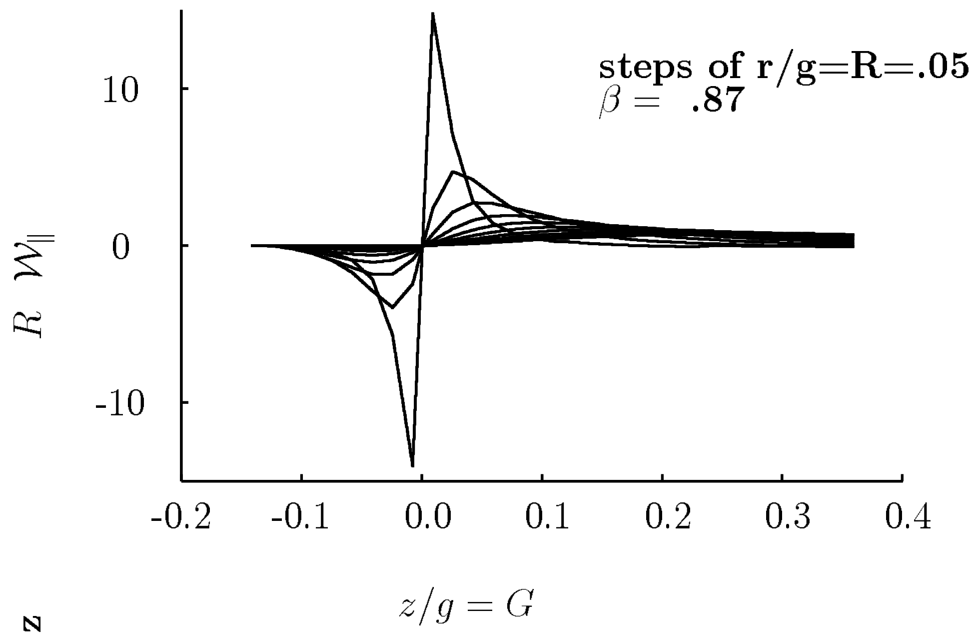
$$
\mathcal{W}_{\parallel\ \sigma_r}(\sigma_r, R, S) \ = \ \int_0^{\infty} \mathcal{W}_{\parallel}(r, S)\ \left\{ \int_0^{2\pi}\ \frac{1}{2\pi}\ e^{\frac{(R - r\cos(\phi))^2 + (r\sin(\phi))^2}{2\ \sigma_r^2}}\ d\phi \right\}\ 2\pi\ r\ dr \tag{11}
$$

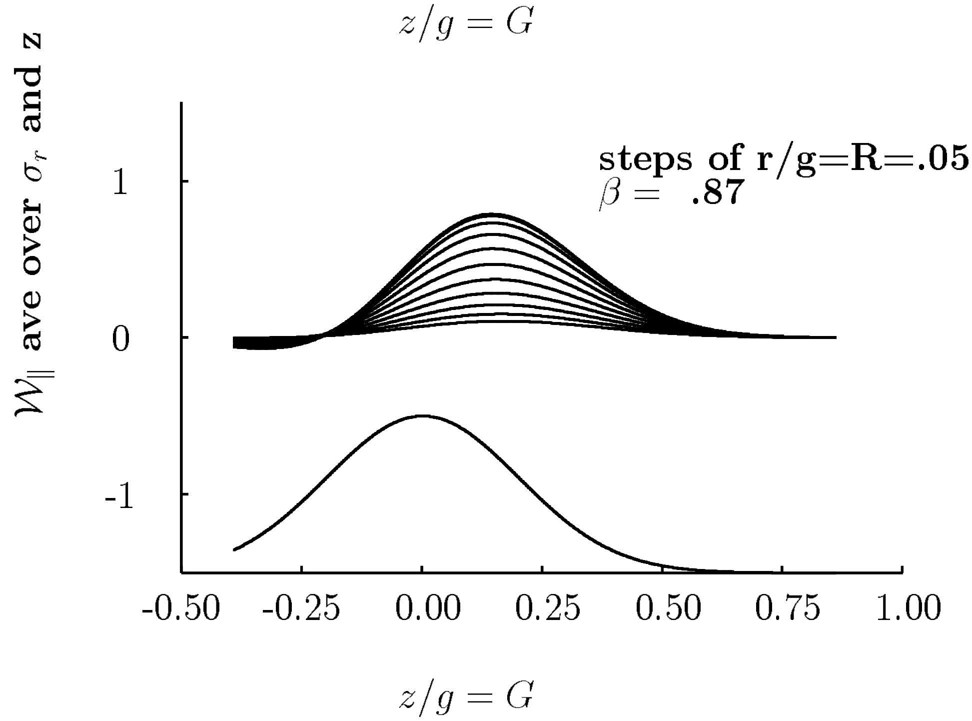
$$\mathcal{W}_{\perp\ \sigma_r}(\sigma_r, R, S) \;=\; \int_0^{\infty} \mathcal{W}_{\perp}(r, S)\ \left\{ \int_0^{2\pi}\ \frac{1}{2\pi}\ e^{\frac{(R - r\cos(\phi))^2 + (r\sin(\phi))^2}{2\ \sigma_r^2}}\ \cos(\phi)\ d\phi \right\}\ 2\pi\ r\ dr \tag{12}$$

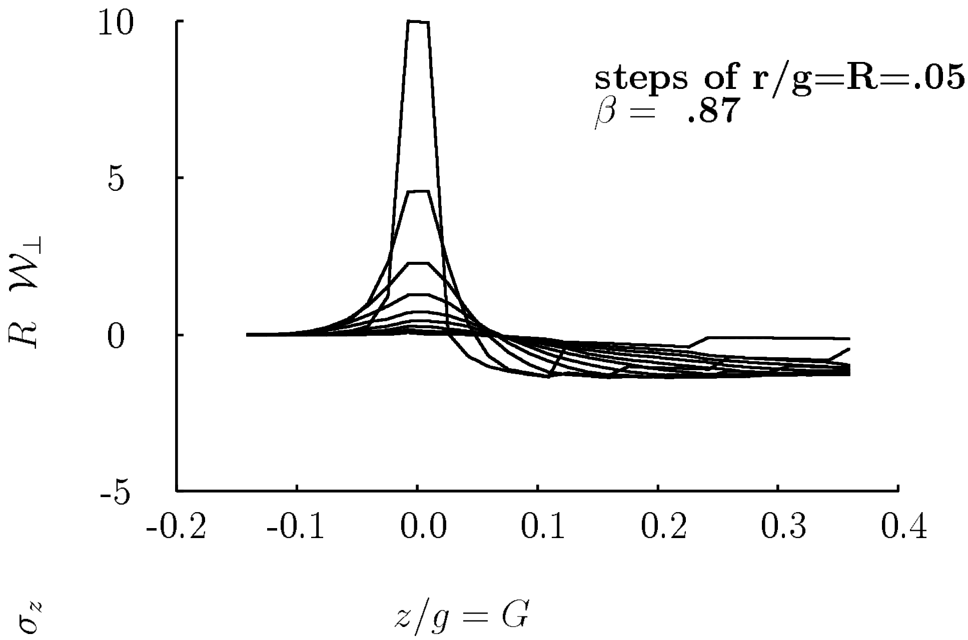
And for either longitudinal or transverse:

$$\mathcal{W}_{\sigma_z,\sigma_r}(\sigma_z, \sigma_r, R, S) \;=\; \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\ \mathcal{W}_{\sigma_r}(\sigma_r, R, z)\ e^{-\frac{(S-z)^2}{2\ \sigma_z^2}}\ dz \tag{13}$$

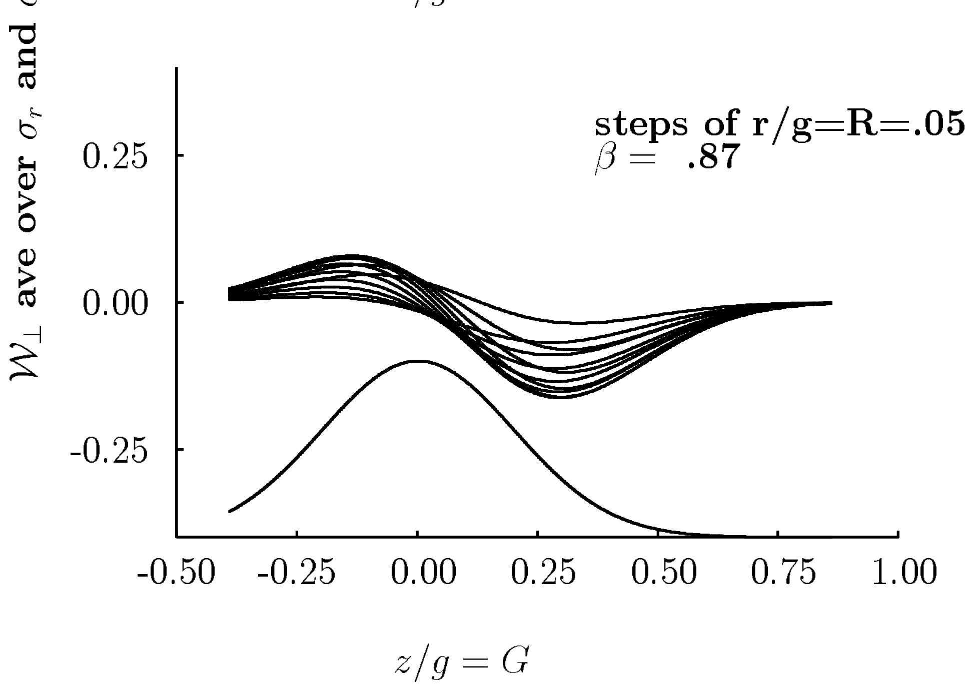
The following wakes were calculated with $\beta = 0.87$, $\sigma_r/g = 0.2$, and $\sigma_z/g = 0.2$:





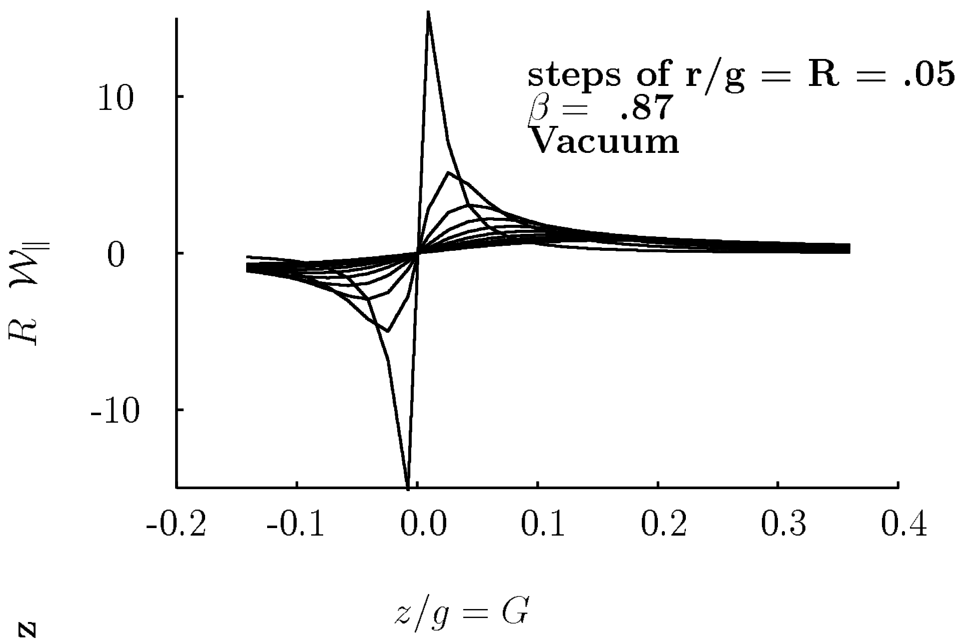


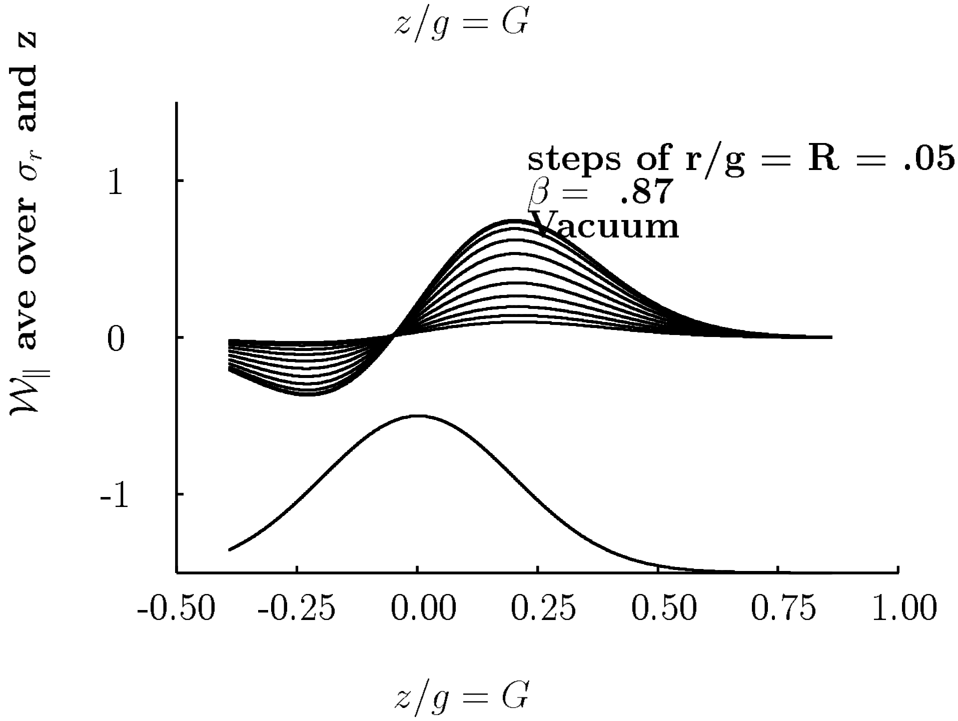
steps of r/g=R=.05
$\beta =$ .87
$R\ \mathcal{W}_\perp$
10
5
0
-5
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
$z/g = G$

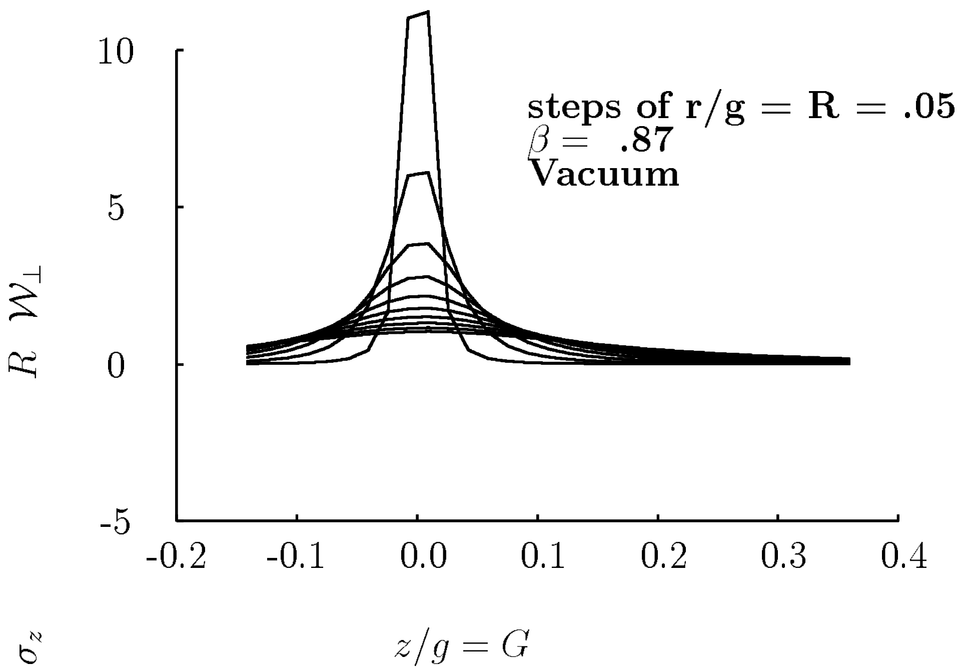


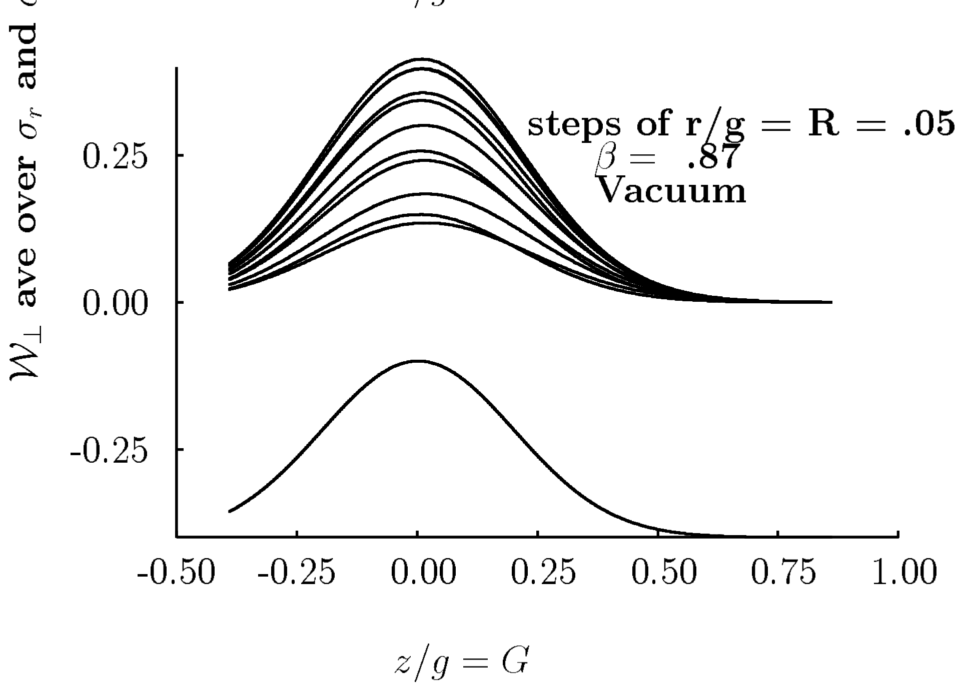
steps of r/g=R=.05
$\beta =$ .87
$\mathcal{W}_\perp$ ave over σ_r and σ_z
0.25
0.00
-0.25
-0.50
-0.25
0.00
0.25
0.50
0.75
1.00
$z/g = G$

The following plots are for the same parameters as above, but calculted for the vacuum case; i.e. no walls of any kind. It will be noted that the transverse wakes in vacuum are about twice those with the foils.






10
5
0
-5
$R\ \mathcal{W}_\perp$
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
steps of r/g = R = .05
$\beta =$ .87
Vacuum
$z/g = G$


$\mathcal{W}_\perp$ ave over σ_r and σ_z
0.25
0.00
-0.25
-0.50
-0.25
0.00
0.25
0.50
0.75
1.00
steps of r/g = R = .05
$\beta =$ .87
Vacuum
$z/g = G$